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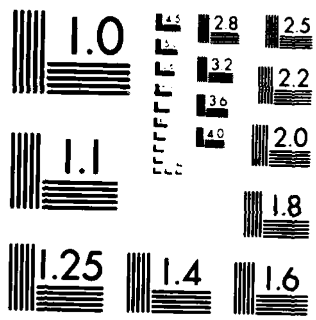
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ABSTRACT

The generalized, bivariate, linear location problem concerns the locating of a linear facility, $x_2=C_0+C_1x_1$, in a two-dimensional Euclidean space such that the p-norm distance taken to the q power is minimized of serving n existing fixed facilities whose location in the two-dimensional Euclidean space is given by (a_{i1},a_{i2}) with $i=1,...,n$. The solution of the generalized, bivariate, linear location problem consists of two subproblems. The first subproblem involves the determination of the point on any linear facility that minimizes the p-norm distance to an individual existing facility. The second subproblem consists of determining the optimal linear facility that minimizes the sum of the q multiples of the p-norm distance from all the existing facilities to the point on the linear facility determined by the previous step.

The lack of convexity of the generalized, bivariate linear location problem prohibits a universal solution technique for all combinations of possible values for p and q. For certain combinations of p's and q's, however, an exact solution can be determined. For example, the

case where the limit is taken as p approaches one from the negative direction and q equals two reduces to simple regression of x_1 on x_2. If the limit is taken as p approaches one from the positive direction and q equals two, the generalized, bivariate, linear location problem becomes simple regression of x_2 on x_1. In general, most combination of p's and q's greater than one can only be solved by a heuristic approximation procedure.

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ON DERIVING AND SOLVING THE GENERALIZED

BIVARIATE, LINEAR LOCATION PROBLEM

by

Joseph William Coleman

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Business Administration

ARIZONA STATE UNIVERSITY

December 1982

ON DERIVING AND SOLVING THE GENERALIZED,

BIVARIATE, LINEAR LOCATION PROBLEM

by

Joseph William Coleman

has been approved

September 1982

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DEDICATION

To my Mother and Father

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I wish to express my sincere appreciation and gratitude to Dr. William Verdini for his guidance and encouragement during the course of this study. His constant interest in my research provided the most fruitful stimulation. I wish also to thank the other members of my committee for the time and energies devoted to this project. The help provided by Dr. Brooks, Dr. Hughes, Dr. Baty and Dr. Smith made the most difficult task a little easier.

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Finally, I wish to thank my wife Florence, and my daughter Joy, for their patience. Only through their encouragement and support was I able to finish.

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CHAPTER I

INTRODUCTION

The generalized, bivariate, linear location problem concerns the placement of a linear facility of the form $x_2 = C_o + C_1x_1$ such that certain criteria dictated by the decision maker are minimized. The linear facility to be located might be considered, for example, from the classical location-theory perspective to be a pipeline connecting a number of factories, a trunk line within a plant connecting work stations, or an interstate highway connecting various cities. A more generic interpretation of the linear facility might involve the estimating (or fixing) of a relationship between output of a machine and power consumed by that machine, sales of a product and the advertising expenditures on that product, or (even in the classical sense) the size of the seeds of the daughter sweet pea plants with respect to the size of the seeds of the mother sweet pea plants (David, 1978).

The critical aspect of the generalized, bivariate, linear location problem concerns the criteria selected by the decision maker to be minimized. If a linear facility cannot pass exactly through each data point, the person performing the location decision must determine which distance should be minimized. Should the objective function of the minimization problem consider just the distance with respect to one of the variables, or should the distance to be minimized be some composite distance with respect to both variables? In addition, the decision maker must determine the relative importance of each of the existing facilities with respect to the distance being minimized. For example, should the importance of an existing facility in the minimization process be linearly proportional to the distance to the linear facility, or should outliers have more or less importance?

Early contributors to point-location literature favored the Euclidean norm of modeling distances in the minimization process. Later, the rectangular norm was used for approximating distances when movement was restricted to a network which was basically a rectangular grid (Morris, 1981). Both norms are special cases of the p-norm distance function,

$$\ell_{pi}(x) = [\sum_{t=1}^{N} |x_t - a_{it}|^p]^{1/p}, \; p \geq 1,$$

where x and a_i are points in N-dimensional space. Love and Morris (1972), considering the point-location problem, found that p-norm distances raised to the power of p = 1.69

and 1.78 were better models for interstate road distances than were the classical rectilinear, $p = 1$, and Euclidean, $p = 2$, distances. Cooper (1968), also working with the point-location model, observed that the weighting of various data points, regardless of the p-norm distance used, should not be necessarily directly proportional to the distance from the point to the linear facility. Depending upon the situation, the p-norm distance taken to powers other than one may more accurately model the situation in terms of costs. Considering the location (or estimation) of linear models, the least squares procedure used in regression minimizes the sum of the squared vertical distances between the data points and the equation $x_2 = C_o + C_1 x_1$ (Neter & Wasserman, 1974); and the least sum procedure minimizes the sum of the absolute values of the same distances.

The generalized, bivariate, linear location problem allows the decision makers first to select the p-norm distance taken to any power q that will best satisfy the needs of their analysis. The linear facility can then be located using the data with respect to the predetermined values of p and q.

Statement of the Problem

The generalized, bivariate, linear location problem can be stated in the following manner: Given n existing fixed facilities i whose locations in a two-dimensional Euclidean space are given by (a_{i1}, a_{i2}) with $i = 1, \ldots, n$; locate a linear facility, $x_2 = C_o + C_1x_1$, in the two-dimensional Euclidean space such that the total cost of serving the n fixed facilities i is minimized. Assume that cost is proportional to distance so that the cost of serving facility i is equal to $W_i \ell^q_{pi}$ where W_i is a weighting constant which transforms distance into cost and ℓ^q_{pi} is the p-norm distance taken to the q power from facility i to the linear facility which is to be located.

More specifically, locating a linear facility consists of two subproblems: (1) determine the point (x_{i1}, x_{i2}) on the linear facility closest to the existing facility i, and (2) determine estimates for the linear parameters, C_o and C_1.

Formally, the modified distance function, $\ell^q_{pi}(C_o, C_1)$, is given by

$$\ell^q_{pi}(C_o, C_1) = [\sum_{t=1}^{2} |a_{it} - x_{it}|^p]^{q/p}$$

subject to

$$C_o + C_1x_{i1} = x_{i2},$$

$$p \geq 1, \text{ and}$$

$$q > 0$$

where x_{it} is the tth coordinate of a point (x_{i1}, x_{i2}) on the linear facility, $x_2 = C_o + C_1x_1$, that minimizes the p-norm distance raised to the q power from the ith existing facility to the linear facility to be located. The preceding definition is similar to the work of Morris (1981) with the addition of the linear constraint.

Using the preceding definition, the formal statement for the linear Weber problem would be

$$\text{Minimize } C\ell_p^q(C_o, C_1) = \sum_{i=1}^{n} W_i \ell_{pi}^q(C_o, C_1)$$

where

C_o, C_1 are the intercept and slope parameters of the linear facility to be determined,

W_i is the weighting constant which transforms distance into costs, and

$\ell_{pi}^q(C_o, C_1)$ is the minimum p-norm distance from the point (a_{i1}, a_{i2}) to the point (x_{i1}, x_{i2}) on the line, $x_2 = C_o + C_1x_1$, taken to the power q.

Review of Related Literature

The subject of generalized, bivariate, linear location is a child with many fathers. If the problem of locating a "linear" facility is considered an extension of the point location area, then the roots of the linear location problem may be traced to Fermat in the early 17th century (Wesolowsky, 1973). If the linear location problem is considered a generalization of the least square estimation procedure, then the heritage of the linear location problem lies in the earliest works of least square estimation done by Gauss in the late 18th century (Eisenhart, 1978). To develop a proper perspective regarding the linear location problem, therefore, consideration must be given to both location theory and least sum regression theory.

Location Theory

Fermat first stated the point location problem in the following manner: "Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is a minimum" (Wesolowsky, 1973, p. 96). Torricelli found the point that minimized the sum of its distances to the points on a triangle to be "at the intersection of the circles which circumscribe the equilateral triangles constructed on the sides of and outside the triangle formed by the three given points" (Verdini, 1976, p. 1). From the work of Cavalieri, Simpson, and Heinen, Jacob Steiner posed and solved the Steiner problem.

> If all angles of the triangle formed by the given points are less than 120°, then the minimizing point is found at the point where three lines, one from

> each given point, intersect at 120°. If one of the angles of the given triangle is greater than or equal to 120°, the vertex of this angle is the minimizing point. (Verdini, 1976, p. 1)

The distances referred to in these solutions were all Euclidean distances raised to the first power ($p = 2$, $q = 1$), and the points were all of equal weight. Simpson, in 1750, was the first to generalize Fermat's problem by investigating the sum of weighted distances (Wesolowsky, 1973).

In 1909 Alfred Weber published his pioneering work in special economics, Über den Standort der Industrien. He posed the problem of placing a factory that produced one output manufactured using two raw inputs from separate localized areas and subsequently sold at one distant market. Weber assigned the three elements i (i.e., one product and two raw inputs) weights u_i depending upon the weight of the element divided by the weight of the product. With this system the weight of the product was defined as one. Weber further defined the transportation rates for each product as r_i. Thus, "Weber termed 'ideal weights,' or weights of elements adjusted for differential transportation rates" (Kuhn & Kuenne, 1962, p. 22) to be

$$W_i = u_i r_i.$$

The total transportation cost, $C\ell_p(x_1, x_2)$ to be minimized is the following:

$$C\ell_2(x_1, x_2) = \sum_{i=1}^{n} W_i \ell_{2i}$$

where ℓ_{2i} is the Euclidean distance from each facility that supplies the raw inputs or receives the final product (a_{i1}, a_{i2}) to the plant to be located (x_1, x_2). The Euclidean distance ℓ_{2i} is mathematically defined as $\ell_{2i} = [(x_1 - a_{i1})^2 + (x_2 = a_{i2})^2]^{1/2}$ (Kuhn & Kuenne, 1962).

The earliest solutions to the Weberian problem were either geometrical or physical in nature. Such models as the weighted triangle (Dean, 1938), Launhart's pole principle (1885), isodapones (Palander, 1935), and the Varignon's frame (Isard, 1956) provided exact solutions, although they were quite cumbersome and limited in the size of problem that they could solve (Kuhn & Kuenne, 1962).

Weiszfeld was the first to discover an iterative technique to solve the original Weberian problem. The technique of iteratively solving two simultaneous equations, first published in 1937, was subsequently rediscovered in the fifties and sixties by various authors (Cooper, 1963; Kuhn & Kuenne, 1962; Michle, 1958). To minimize the cost function, the first derivatives are taken with respect to x_1 and x_2, and the results are set equal to zero. This process yields

$$\frac{\partial C\ell_2(X)}{\partial x_1} = \sum_{i=1}^{n} \frac{W_i(x_1 - a_{i1})}{\ell_{2i}(x_1, x_2)} = 0,$$

$$\frac{\partial C\ell_2(X)}{\partial x_2} = \sum_{i=1}^{n} \frac{W_i(x_2 - a_{i1})}{\ell_{2i}(x_1, x_2)} = 0.$$

Rearranging terms produces the following equations:

$$x_1 = \frac{\sum_{i=1}^{n} \frac{W_i a_{i1}}{\ell_{2i}(x_1, x_2)}}{\sum_{i=1}^{n} \frac{W_i}{\ell_{2i}(x_1, x_2)}},$$

$$x_2 = \frac{\sum_{i=1}^{n} \frac{W_i a_{i2}}{\ell_{2i}(x_1, x_2)}}{\sum_{i=1}^{n} \frac{W_i}{\ell_{2i}(x_1, x_2)}}.$$

An iterative procedure is required since $x_1 = f(x_1, x_2)$ and $x_2 = f(x_1, x_2)$. The weighted mean, the center of gravity, was a convenient starting point for the iteration. The preceding function was found to be convex since it was the sum of convex functions, and the iterative procedure was assumed to converge on the global minimum if no iteration resulted in x_1 and x_2 being equal to the coordinates of one of the existing facilities.

Cooper (1968) generalized the Weber problem to minimize the Euclidean distance raised to a power q where q was greater than zero. Love and Morris (1972) extended the Weber problem to include p-norm distances, but Morris (1981) combined these two formulations into the generalized Weber problem which can be stated in the following manner for Euclidean two space:

$$\text{Minimize } C\ell_p^q(x_1, x_2) = \sum_{i=1}^{n} W_i \ell_{pi}^q(x_1, x_2)$$

where

$$\ell_{pi}^q(x_1, x_2) = [\sum_{t=1}^{2} |x_{it} - a_{it}|^p]^{q/p}.$$

Inspired by the work of Love and Morris (1972), Wesolowsky and Love (1972), Eyster, White, and Wierwille (1973), and Love (1969), Verdini (1976) developed and proved convergence of a generalized hyperbolic approximating function that eliminated the difficulties in differentiating the generalized Weber problem for q = 1 and p = 1 or 2. The approximating function, $L_{pi}^q(x_1, x_2)$, took the following form:

$$L_{pi}^q(x_1, x_2) = \{\sum_{t=1}^{2} [(x_{it} - a_{it})^2 + \varepsilon]^{p/2}\}^{q/p}$$

where ε was a strictly positive smoothing constant. Morris and Verdini (1979) extended the application of the hyperbolic approximating function by proving convergence and convexity for $q = 1$ and $p \geq 1$, and Morris (1981) further generalized the convexity and convergence proofs for $q \geq 1$ and $p > 0$. As Love and Morris (1978) proved for $p = 2$ and Morris (1981) proved for $p > 0$, however, the Weber problem is not convex or concave for $q < 1$. Local minima exist at each existing facility.

Location theory was extended from the point perspective to the linear perspective by MacKinnon and Barber (1972) when they developed an algorithm that positioned a linear facility that minimized the Euclidean distance ($p = 2$, $q = 1$). They used a heuristic method to search for the optimal slope of the linear facility between bounds of the slope of the regression lines of x_2 on x_1 and x_1 on x_2.

Wesolowsky (1975) determined an exact, although not necessarily an unique, solution to the linear location problem for Euclidean distance ($p = 2$, $q = 1$) by rotating the original coordinate axes until the optimal slope of the linear facility was parallel to the transformed axes. The points (x_{i1}, x_{i2}) where $i = 1, \ldots, n$ were transformed into polar coordinates $(x_{i1\rho}, x_{i2\theta})$ in the following manner:

$$x_{i2\theta} = x_{i2} \cos \theta + x_{i1} \sin \theta,$$

$$x_{i1\theta} = -x_{i2} \sin \theta + x_{i1} \cos \theta.$$

Thus, with the knowledge that the line that minimizes the $p = 2$, $q = 1$ distance is the median line (Rice & White, 1964), Wesolowsky found the value of θ for the line $x_{i2\theta} = Z_\theta$ in the transformed axes that minimized the following equation:

$$C\ell_2^1(\theta) = \sum_{i=1}^{n} W_{i\theta} |Z_\theta - x_{i1\theta}|.$$

Once the value of θ was found, C_1 was equal to tan θ; and C_o was the median of $x_{i2\theta}$.

Morris and Norback (1980) simplified Wesolowsky's procedure by recognizing that the linear facility for $q = 1$ was the median line and that at least one solution would pass through at least two existing facilities i. Morris and Norback also extended the applicability of their procedure to $0 < q \leq 1$ distances. The authors pointed out, however, that the convexity condition for $q - 1$ was not present for $1 < q < \infty$, and that an alternate heuristic technique must be used for solving problems with values of $q > 1$.

Least Sum Regression Theory

The least square estimation procedure, which is the basis for most linear regression applications, was first published by Gauss in 1794. His treatment of the location of a function in the form $y = \alpha + \beta X$ which minimized the squared error from a set of points n where $n > 2$ was inspired by John Neinrick Lambert's approach to the same problem by using a moving average (Eisenhart, 1978). Francis Galton indirectly coined the term "regression" in his 1885 publication, Natural Inheritance, when he studied sweet pea plants. Galton noticed that the size of the seeds of daughter plants seemed to be related to the size of the seeds of the mother plant, but that they also appeared to "revert" to the mean. The term "revert" was soon replaced by "regress" (David, 1978).

A relatively short time after Gauss, in the 1820s Fourier first proposed a method of minimizing the sum of absolute deviations instead of minimizing the sum of the squared deviations as used in regression. Fourier recommended an iterative technique that he developed which was similar to the simplex method (Fisher, 1961). Later in the century, Edgeworth, receiving inspiration from the works of Laplace, recognized that the least squares procedure was highly susceptible to outliers and was, therefore, inferior to least sum estimators (Barrodale, 1968).

This observation was later supported by the studies of Barrodale (1968) and Forsythe (1972). The algorithm proposed by Edgeworth (Rhodes, 1930), as well as more recent algorithms developed by Rhodes (1930), Singleton (1940), and Karst (1958), were either limited to two dimensions or became unwielding as the number of dimensions increased (Fisher, 1961).

Each of the preceding authors recognized that minimizing the sum of the absolute deviations did not necessarily produce a unique solution. Rice and White (1964) pointed out that the y-intercept (C_o) was the median of $a_{i2} - C_i a_{i1}$; and, therefore, the best least sum estimator would be the median line. Sposito and Smith (1976), expanding on the work of Appa and Smith (1973), provided the following conditions for the location of a hyperplane of R dimensions from n observations which minimized the sum of absolute deviations:

1. At least one hyperplane giving the minimum sum of absolute deviations passes through R of the n points.

2. Under the assumption that no set of R = 1 observations lies on one hyperplane in R dimensions, if one denotes n_1 as the number of points above a certain hyperplane and n_2 as the number of points below this hyperplane, then the hyperplane cannot minimize the sum of absolute deviations unless $|n_1 - n_2| \leq R$.

3. If the number of observations, n, is odd, then any hyperplane that minimizes the sum of absolute deviations passes through at least one observation point.

4. For a given set of observations, if multiple hyperplanes exist that minimize the sum of absolute deviations, then any convex combination of these optimal hyperplanes is also optimal.

Charnes, Cooper, and Ferguson (1955) demonstrated the applicability of simplex as a means of minimizing the sum of absolute deviations. In their procedure they developed a linear programming problem that consisted of n constraints where n was equal to the number of data points and that contained an objective function that minimized the sum of the non-negative deviations for each constraint. Wagner (1959) modified the Charnes et al. formulation by reducing the number of non-negative deviations to two variables to solve the Chebyshev criterion: Minimize the maximum deviation. Various linear programming formulations of each minimization criterion have been addressed by many authors. Among these are Barrodale and Roberts (1973), Barrodale and Young (1966), Glahe and Hunt (1970), Fisher (1961), Sielken and Hartley (1973), and Narula and Wellington (1977). Forsythe (1972) extended the minimum sum concept to minimize the sum of the deviations taken to powers between zero and infinity by the use of gradient search techniques.

The preceding statistically oriented models (i.e., least square and least sum regression) were designed to minimize the vertical distance between the observation points and the linear equation to be estimated. Using the linear location theory notation, the assumption in these models was that the minimum distance between the existing facility i located at a_{i1}, a_{i2} and the linear facility, $x_2 = C_o + C_1x_1$, was always equal to $a_{i2} - C_o - C_1a_{i1}$. An exception to this concept exists in the field of statistical estimation. With the technique of principal components, the Euclidean distance raised to the second power ($p = 2$, $q = 2$) is minimized by finding a linear composite of the original variables such that the sum of the squared perpendicular distances from the observations to the linear composite are minimized (Green, 1978).

Summary

In summary, many solution techniques are available to solve individual cases of the generalized, bivariate, linear location problem.

For regression, which minimizes the vertical distances, the least squares procedure is used to minimize these distances raised to the second power; linear to powers between zero and infinity by the use of gradient search techniques.

The preceding statistically oriented models (i.e., least square and least sum regression) were designed to minimize the vertical distance between the observation points and the linear equation to be estimated. Using the linear location theory notation, the assumption in these models was that the minimum distance between the existing facility i located at a_{i1}, a_{i2} and the linear facility, $x_2 = C_o + C_1x_1$, was always equal to $a_{i2} - C_o - C_1a_{i1}$. An exception to this concept exists in the field of statistical estimation. With the technique of principal components, the Euclidean distance raised to the second power ($p = 2$, $q = 2$) is minimized by finding a linear composite of the original variables such that the sum of the squared perpendicular distances from the observations to the linear composite are minimized (Green, 1978).

Summary

In summary, many solution techniques are available to solve individual cases of the generalized, bivariate, linear location problem.

For regression, which minimizes the vertical distances, the least squares procedure is used to minimize these distances raised to the second power; linear programming formulations are used to minimize these distances raised to the first power and raised to infinity; and gradient search procedures are used for these distances raised to any powers between zero and infinity. For minimizing the Euclidean distance ($p = 2$), Wesolowsky's procedure minimizes $p = 2$ distances raised to the first power; Morris and Norback's modification of Wesolowsky's procedure, in addition to being used to minimize the absolute distances, can be used for $p = 2$ distances raised to powers between 0 and 1; and the principal components method minimizes the square of $p = 2$ distances.

Organizational Plan of the Study

An analysis of the generalized, bivariate, linear location problem is presented in this study.

Two methods for deriving the point (x_{i1}, x_{i2}) on the linear facility that minimizes the p-norm distance taken to the q power to the existing facility are discussed in Chapter II. In addition, solution properties and a physical interpretation of the p-norm distance within the generalized, bivariate, linear location problem are presented.

Contained in Chapter III is a presentation of the formulation of the modified distance function, $\ell^{q}_{pi}(C_o, C_1)$. A modification of the hyperbolic approximation technique of Morris and Verdini (1979) is used to investigate the convexity properties of $\ell^q_{pi}(C_o, C_1)$. Also, the Weber formulation of the generalized, bivariate, linear location problem based on the modified hyperbolic approximation function along with its convexity properties are discussed.

Analytical solutions of the Weber formulation of the generalized, bivariate, linear location problem for certain values of p and q are presented in Chapter IV. Also, in Chapter IV a heuristic solution for the generalized, bivariate, linear location problem for values of p and q not analytically solvable is presented.

Included in Chapter V are a summary and recommendations for further research. Following Chapter V is the bibliography which concludes this study.

CHAPTER II

WEBER REFORMULATION: POINT DETERMINATION

Consideration must be given to procedures for restating the Weber formulation of the generalized, bivariate, linear location problem as presented in Chapter I in a format that can be used to identify the unknowns, C_o and C_1, of the linear location facility, $x_2 = C_o + C_1x_1$.

Locating a linear facility is a two-step process: (1) determine the point (x_{i1}, x_{i2}) on the linear facility "closest" to the existing facility i, and (2) determine estimates for the linear parameters C_o and C_1. In this chapter step 1 is discussed. Specifically, the point given by the coordinates (x_{i1}, x_{i2}) that minimizes the modified p-norm distance from the existing facility to the linear facility is determined.

As presented in Chapter I, the modified distance function, $\ell^q_{pi}(C_o, C_1)$, is as follows:

$$\text{Minimize } \ell^q_{pi}(C_o, C_1) = [\sum_{t=1}^{2} |a_{it} - x_{it}|^p]^{q/p} \tag{2.1}$$

subject to

$$C_o + C_1x_{i1} = x_{i2},$$

$p \geq 1$, and

$q > 0$.

As stated, the modified distance function, $\ell^q_{pi}(C_o, C_1)$, contains four unknowns. The obvious unknowns, C_o and C_1, are the parameters needed to place the linear facility. C_o and C_1 cannot be found, however, until x_{i1} and x_{i2}, the points on the linear facility that minimize the p-norm distance taken to the q power from the linear facility to the existing facility i, are determined. x_{i1} and x_{i2} cannot be determined, however, until C_o and C_1 are found.

In regression, which minimizes the sum of the squared vertical distances, the values of x_{i1} and x_{i2} can be stated as functions of C_o, C_1, a_{i1}, and a_{i2} as follows:

$$x_{i1} = a_{i1},$$

$$x_{i2} = C_o + C_1 a_{i1}.$$

For the Euclidean ($p = 2$) distances the values of x_{i1} and x_{i2} which minimize the modified distance function correspond to the perpendicular distances from the linear facility to the existing facility i. In general, however, the values of x_{i1} and x_{i2} that minimize the p-norm distance taken to the q power are not on the vertical or the perpendicular and must be analytically determined.

The first step in solving the preceding circular condition is to determine a general expression for x_{i1} and x_{i2} as a function of C_o, C_1, a_{i1}, and a_{i2} for any p-norm distance taken to the q power. At this stage of the analysis C_o and C_1 are considered given, and the variables to be optimized are the coordinates x_{i1} and x_{i2}. In other words, given a line, $x_2 = C_o + C_1x_1$, (where C_o and C_1 are known) and a point (a_{i1}, a_{i2}), the coordinates x_{i1} and x_{i2} on the line that minimize the p-norm distance taken to the q power must be found. Mathematically stated, the problem is to solve the following equation:

$$\text{Minimize } \ell^q_{pi}(x_1, x_2) = [\sum_{t=1}^{2} |a_{it} - x_{it}|^p]^{q/p} \qquad (2.2)$$

subject to

$$C_o + C_1x_{i1} = x_{12},$$

$$p \geq 1, \text{ and}$$

$$q > 0.$$

Lagrangian Formulation

One method of finding the point (x_{i1}, x_{i2}) on the linear facility, $x_2 = C_o + C_1x_1$, is to solve the following Lagrangian formulation:

$$L(X, \lambda) = \ell_{pi}^{q}(x_{i1}, x_{i2})$$

$$+ \lambda(x_{i2} - C_1 x_{i1} - C_o). \qquad (2.3)$$

Taking the partial derivatives with respect to x_{i1}, x_{i2}, and λ, the following equations result:

$$\frac{\partial L(X, \lambda)}{\partial x_{i1}} = q|a_{i1} - x_{i1}|^{p-1}(\ell_{pi}^{q})^{1-p/q}(-1)$$

$$* \ \mathrm{sgn}(a_{i1} - x_{i1}) - \lambda C_1, \qquad (2.4)$$

$$\frac{\partial L(X, \lambda)}{\partial x_{i2}} = q|a_{i2} - x_{i2}|^{p-1}(\ell_{pi}^{q})^{1-p/q}(-1)$$

$$* \ \mathrm{sgn}(a_{i2} - x_{i2}) + \lambda, \qquad (2.5)$$

$$\frac{\partial L(X, \lambda)}{\partial \lambda} = x_{i2} - C_1 x_{i1} - C_o \qquad (2.6)$$

where sgn is the signum function with the following properties:

if $D > K$, then $\mathrm{sgn}(D - K) = 1$,

if $D = K$, then $\mathrm{sgn}(D - K) = 0$, and

if $D < K$, then $\mathrm{sgn}(D - K) = -1$.

Equations (2.4), (2.5), and (2.6) must each be set equal to zero and solved simultaneously.

<u>Property 2.1</u>

If $a_{i1} = x_{i1}$ and $a_{i2} = x_{i2}$, then the first-order partial derivatives of $L(X, \lambda)$ are undefined for $p \geq q$.

Proof. Let $a_{i1} = x_{i1}$ and $a_{i2} = x_{i2}$. Then the first-order partial derivatives of $L(X, \lambda)$ with respect to x_{it} (there $t = 1$ or 2) are as follows:

For $p = q$:

$$\frac{\partial L(X, \lambda)}{\partial x_{it}} = q|0|^{p-1}[(0)^p + (0)^p]^{1-1} \operatorname{sgn}(0)(-1).$$

For $p > q$:

$$\frac{\partial L(X, \lambda)}{\partial x_{it}} = q|0|^{p-1} \frac{-\operatorname{sgn}(0)}{[(0)^p + (0)^p]^{p-q/p}}$$

$$= q|0|^{p-1} \operatorname{sgn}(0)(1/0)(-1).$$

Since $[0]^0$ and $1/0$ are undefined, $\partial L(X, \lambda)/\partial x_{it}$ is undefined. The first-order partial derivatives of $L(X, \lambda)$ are undefined for $p \geq q$.

Property 2.2

If $a_{it} = x_{it}$ for $t = 1$ or 2, then the first-order partial derivative of $L(X, \lambda)$ with respect to x_{it} is undefined for $-1 \leq p \leq 1$.

Proof. Let $a_{it} = x_{it}$ for $t = 1, 2$. Then the first-order partial derivatives of $L(X, \lambda)$ with respect to x_{it} are:

For $p = 1$:

$$\frac{\partial L(X, \lambda)}{\partial x_{it}} = q|0|^{1-1}(\ell_{pi}^{q})^{1-1/q} \operatorname{sgn}(0)(-1) = |0|^0.$$

For $-1 \leq p \leq 1$:

$$\frac{\partial L(X, \lambda)}{\partial x_{it}} = \frac{-q(\ell_{pi}^{q})^{1-1/q} \operatorname{sgn}(0)}{|0|^{(1-p)}} = \frac{-1}{|0|^{(1-p)}}.$$

Again, since $|0|^0$ and $1/0$ are undefined, $\partial L(X, \lambda)/\partial x_{it}$ is undefined for $-1 \leq p \leq 1$.

Property 2.3

If $a_{i1} = x_{i1}$ and $a_{i2} = x_{i2}$, then the rth-order partial derivatives of $L(X, \lambda)$ are undefined for $p \geq q/r$ and $-r \leq p \leq r$.

Proof. Using the same arguments as those used in Property 2.2, the values for the rth-order partial derivatives of $L(X, \lambda)$ for the ranges of p stated previously are equal to 0^0 or $1/0$ and, therefore, are undefined.

Alternate Formulation

The computational problems associated with the signum function and the undefined nature of the Lagrangian function, $L(X, \lambda)$, require that an alternate formulation of the problem be used to determine the point (x_{i1}, x_{i2}) on the linear facility, $x_2 = C_o + C_1x_1$, closest to the existing facility i.

Again, recognizing that locating a linear facility is actually a two-step process (i.e., the first to determine the point (x_{i1}, x_{i2}) on the linear facility closest to the existing facility i, and the second to determine estimates for the linear parameters C_o and C_1), the problem is divided into two sections. Step 1 (i.e., finding the point (x_{i1}, x_{i2}) as a function of C_o, C_1, a_{i1}, and a_{i2} on any linear facility, $x_2 = C_o + C_1x_1$, closest to the existing facility i located at (a_{i1}, a_{i2}) is determined here.

Since the purpose is to determine the point (x_{i1}, x_{i2}) on a given linear facility, the values of C_o and C_1 can be assumed to be known at this stage of the analysis. In theory, however, the process of determining x_{i1} and x_{i2} would be repeated an infinite number of times for each possible combination of values for C_o and C_1.

Since C_o and C_1 are assumed known at this stage of the analysis, whether an existing facility i lies on the given linear facility is known. If this is the case, then $x_{i1} = a_{i1}$ and $x_{i2} = C_o - C_1a_{i1} = a_{i2}$ and the modified distance function ℓ^q_{pi} equals zero. For an existing facility i not on a given linear facility, the Lagrangian formulation for the modified distance function as a function of x_{i1} and x_{i2} is similar to Equation (2.3):

$$L(X, \lambda) = \ell^q_{pi}(x_{i1}, x_{i2}) + \lambda(x_{i2} - C_1x_{i1} - C_o). \qquad (2.7)$$

For points on the linear facility, $L(X, \lambda) = 0$.

Substituting Equation (2.2) for $\ell^{q}_{pi}(x_{i1}, x_{i2})$, the Lagrangian formulation becomes:

$$L(X, \lambda) = [|a_{i1} - x_{i1}|^{p} + |a_{i2} - x_{i2}|^{p}]^{q/p}$$
$$+ \lambda(x_{i2} - C_{1}x_{i1} - C_{o}). \qquad (2.8)$$

The computational problems associated with the absolute value function within the modified distance function are easily eliminated by squaring and taking the square root of the expressions within the absolute value functions. After performing this operation and setting the resulting relationships into a Lagrangian formulation, the following formula results:

$$\text{Minimize } L(X, \lambda) = \{[(a_{i1} - x_{i1})^{2}]^{p/2}$$
$$+ [(a_{i2} - x_{i2})]^{p/2}\}^{q/p}$$
$$+ \lambda (x_{i2} - C_{1}x_{i1} - C_{o}) \qquad (2.9)$$

Taking the partial derivatives with respect to each variable, the following equations result:

$$\frac{\partial L(X, \lambda)}{\partial x_{i1}} = -q\{[(a_{i1} - x_{i1})^{2}]^{p/2}$$
$$+ [(a_{i2} - x_{i2})^{2}]^{p/2}\}^{(q/p)-1}$$
$$* [(a_{i1} - x_{i1})^{2}]^{(p/2)-1}$$
$$* (a_{i1} - x_{i1}) - \lambda C_{1}, \qquad (2.10)$$

$$\frac{\partial L(X,\ \lambda)}{\partial x_{i2}} = -q\{[(a_{i1} - x_{i1})^2]^{p/2}$$

$$+ [(a_{i2} - x_{i2})^2]^{p/2}\}^{(q/p)-1}$$

$$* [(a_{i2} - x_{i2})^2]^{(p/2)-1}$$

$$* (a_{i2} - x_{i2}) + \lambda, \qquad (2.11)$$

$$\frac{\partial L(X,\ \lambda)}{\partial \lambda} = x_{i2} - C_1 x_{i1} - C_o. \qquad (2.12)$$

Most of the problems associated with the original Lagrangian formulation have been eliminated. The first-order partial derivatives with respect to x_{i1} and x_{i2} now can be computed since the signum function has been eliminated, and the first-order partial derivatives are defined for all values of p and q as long as $a_{i1} \neq x_{i1}$ and $a_{i2} \neq x_{i2}$. This nonequality condition for both variables has been eliminated. One problem, however, still remains. If just one $a_{it} = x_{it}$ (i.e., either $a_{i1} = x_{i1}$ or $a_{i2} = x_{i2}$), then $\partial L(X,\ \lambda)/\partial x_{it}$ is undefined for $p \leq 2$.

Ignoring this property for the moment and recognizing that the following derivation may not be valid for $p \leq 2$, the values of x_{i1} and x_{i2} can be solved by setting Equations (2.10), (2.11), and (2.12) equal to zero and solving simultaneously. Performing these steps and eliminating x_{i2}, the following equation results:

$$(a_{i1} - x_{i1})[(a_{i1} - x_{i1})^2]^{(p/2)-1}$$

$$= -C_1(a_{i2} - C_o - C_1x_{i1})$$

$$* \ [(a_{i2} - C_1x_{i1} - C_o)^2]^{(p/2)-1}. \tag{2.13}$$

Squaring both sides of the equation to ensure that C_1 is positive and taking the (p-1)th root, the following formula is obtained:

$$(a_{i1} - x_{i1})^2 = [C_1{}^2]^{1/(p-1)}$$

$$* \ (a_{i2} - C_o - C_1x_{i1})^2. \tag{2.14}$$

Expanding and collecting terms and then using the quadratic formula, the following values of x_{i1} are obtained:

$$x_{i1} = \left\{\frac{a_{i1} + [(C_1{}^2)^{1/2}]^{1/(p-1)}[a_{i2} - C_o]}{1 - [C_1{}^2]^{p/(p-1)}}\right\}$$

$$* \ \{1 - [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}, \tag{2.15a}$$

$$\left\{\frac{a_{i1} - [(C_1{}^2)^{1/2}]^{1/(p-1)}[a_{i2} - C_o]}{1 - [C_1{}^2]^{p/(p-1)}}\right\}$$

$$* \ \{1 + [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}. \tag{2.15b}$$

Substituting the values of x_{i1} into Equation (2.12), the following values for x_{i2} are obtained:

$$x_{i2} = \frac{C_o + C_1 a_{i1} - [C_1^2]^{p/(p-1)} a_{i2}}{1 - [C_1^2]^{p/(p-1)}}$$

$$+ \frac{[C_1][(C_1^2)^{1/2}]^{1/(p-1)}[a_{i2} - C_o - C_1 a_{i1}]}{1 - [C_1^2]^{p/(p-1)}}, \tag{2.16a}$$

$$\frac{C_o + C_1 a_{i1} - [C_1^2]^{p/(p-1)} a_{i2}}{1 - [C_1^2]^{p/(p-1)}}$$

$$+ \frac{[C_1][(C_1^2)^{1/2}]^{1/(p-1)}[a_{i2} - C_o - C_1 a_{i1}]}{1 - [C_1^2]^{p/(p-1)}}. \tag{2.16b}$$

The pair of solutions for x_{i1} and x_{i2} results from the squaring of C_1 in the solution procedure. This technique is necessary since the slope (C_1) may be negative and the (p-1)th root may be noninteger. Resolving the conflict of two solutions for x_{i1} and x_{i2} is accomplished by substituting each pair of values into Equation (2.13). Performing this action for the first values of x_{i1} and x_{i2}, the following equation results:

$$-[(C_1^2)^{1/2}] = -C_1. \tag{2.17}$$

The term on the left side of the equal sign is always less than or equal to zero, and the only way for the equation to be equal is for $C_1 \geq 0$. Thus, the first values of x_{i1} and x_{i2} are valid solutions if and only if $C_1 \geq 0$.

Likewise, performing the same action with the second values of x_{i1} and x_{i2}, the following equation results:

$$[(C_1^2)^{1/2}] = -C_1. \tag{2.18}$$

In this case the term on the left side of the equation sign is always greater than or equal to zero. Thus, C_1 must be less than or equal to zero, and the second values of x_{i1} and x_{i2} are valid solutions if and only if $c_1 \leq 0$.

Solution Properties of x_{i1} and x_{i2}

The following properties and associated proofs substantiate the assertion that the point (x_{i1}, x_{i2}) minimizes the modified p-norm distance from the existing facility i to the linear facility.

Property 2.4

For values of $p \geq 1$ and $q \geq 0$ the point (x_{i1}, x_{i2}) minimizes the p-norm distance taken to the q power from the existing facility i to the linear facility, $x_2 = C_o + C_1x_1$.

Proof. Substituting $C_o + C_1X_{i1}$ for X_{i2} in Equation (2.9) and remembering that $X_{i2} - C_1X_{i1} - C_o = 0$, the following equation results:

$$L(x_{i1}) = \{[(a_{i1} - x_{i1})^2]^{p/2} + [(a_{i2} - C_o - C_1x_{i1})^2]^{p/2}\}^{q/p}. \quad (2.19$$

Taking the total derivatives with respect to x_{i1}, the following result is obtained:

$$
\begin{aligned}
\frac{dL(x_{i1})}{dx_{i1}} = {} & -q\{[(a_{i1} - x_{i1})^2]^{p/2} \\
& + [(a_{i2} - C_o - C_1 x_{i1})^2]^{p/2}\}^{(q/p)-1} \\
& * \underline{\{[(a_{i1} - x_{i1})^2]^{(p/2)-1}(a_{i1} - x_{i1})} \\
& + \underline{C_o[(a_{i1} - C_o - C_1 x_{i1})^2]^{(p/2)-1}} \\
& * \underline{(a_{i2} - C_o - C_1 x_{i1})}\}.
\end{aligned}
\tag{2.20}
$$

Notice that the portion of Equation (2.20) underlined would be equivalent to Equation (2.13) if the first derivative were set equal to zero. Taking the second derivative of $L(x_{i1})$ with respect to x_{i1} produces the following result:

$$
\begin{aligned}
\frac{d^2L(x_{i1})}{dx_{i1}^{\,2}} = {} & q\{[(a_{i1} - x_{i1})^2]^{p/2} \\
& + [(a_{i2} - C_o - C_1 x_{i1})^2]^{p/2}\}^{(q/p)-2} \\
& * \underline{[(q - p)\{[(a_{i1} - x_{i1})^2]^{(p/2)-1}(a_{i1} - x_{i1})} \\
& + \underline{C_1[(a_{i1} - C_o - C_1 x_{i1})^2]^{(p/2)-1}} \\
& * \underline{(a_{i1} - C_o - C_1 x_{i1})\}^2}
\end{aligned}
$$

$$+ (p - 1)\{[(a_{i1} - x_{i1})^2]^{p/2}$$

$$+ [(a_{i2} - C_o - C_1x_{i1})^2]^{p/2}\}$$

$$* \{[(a_{i1} - x_{i1})^2]^{(p/2)-1}$$

$$+ C_1{}^2[(a_{i2} - C_o - C_1x_{i1})^2]^{(p/2)-1}\}]. \quad (2.21)$$

Notice that the portion of Equation (2.21) that is underlined is equal to zero for solutions derived, and all of the remaining terms are strictly positive except those containing q and (p-1). Thus, the second derivative of $L(x_{i1})$ is strictly positive as long as q and (p-1) are greater than or equal to zero. The second derivative is also strictly positive if $q \geq 0$ and $p \geq 1$.

<u>Property 2.5</u>

The point (x_{i1}, x_{i2}) on the linear facility, $x_2 = C_o + C_1x_1$, that minimizes the modified distance function $\ell^q_{pi}(X)$ is independent of q as observed from Equations (2.15a), (2.15b), (2.16a), and (2.16b).

<u>Property 2.6</u>

If the existing facility i is on the linear facility, $x_2 = C_o + C_1x_1$; then $x_{i1} = a_{i1}$, and $x_{i2} = a_{i2}$.

<u>Proof</u>. (Since the proofs for $x_{i1} = a_{i1}$ for Equations (2.15a) and (2.15b) and the proofs for $x_{i2} = a_{i2}$

for Equations (2.16a) and (2.16b) are similar, only $x_{i1} = a_{i1}$ for Equation (2.15a) is proven here.) If the existing facility i is on the linear facility, then $a_{i2} = C_o + C_1 a_{i1}$. Substituting this relationship into Equation (2.15a), the following equation results:

$$x_{i1} = \{\frac{a_{i1} + [(C_1{}^2)^{1/2}]^{1/(p-1)}[C_o + C_1 a_{i1} - C_o]}{1 - [C_1{}^2]^{p/(p-1)}}\}$$

$$* \quad \{1 - [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}. \qquad (2.22)$$

Since C_1 must be greater than zero for Equation (2.15a) to be a solution to Equation (2.13), the following operation can be performed:

$$x_{i1} = \frac{a_{i1}\{1 + [(C_1{}^2)^{1/2}]^{p/(p-1)}\}\{1 - [(C_1{}^2)^{1/2}]^{p/(p-1)}\}}{1 - [C_1{}^2]^{p/(p-1)}},$$

$$= a_{i1}.$$

The following properties eliminate the problem associated with the first-order partial derivatives of the modified distance function.

Property 2.7

The slope of the line connecting the existing facility i to the point (x_{i1}, x_{i2}) on the linear facility, $x_2 = C_o + C_1 x_1$, that minimizes the p-norm distance taken to the q power is equal to $-1/[(C_1{}^2)^{1/2}]^{1/(p-1)}$ if $C_1 \geq 0$ and equal to $1/[(C_1{}^2)^{1/2}]^{1/(p-1)}$ if $C_1 \leq 0$.

Proof. As previously stated, the coordinates of the point on the linear facility that minimize the p-norm distance taken to the q power from the linear facility, $x_2 = C_o + C_1x_1$, to the existing facility i when $C_1 \geq 0$ are given by Equations (2.15a) and (2.16a). Substituting these values into the definition of the slope, the following equation results:

$$\text{Slope} = \frac{\Delta X_2}{\Delta X_1} = \frac{a_{i2} - x_{i2}}{a_{i1} - x_{i1}} = \frac{-1}{[(C_1{}^2)^{1/2}]^{1/p-1}}.$$

For $C_1 \leq 0$ the optimal values for x_{i1} and x_{i2} are given by Equation (2.15a) and Equation (2.16b), respectively. Substituting these values into the definition of the slope, the following equation results:

$$\text{Slope} = \frac{\Delta X_2}{\Delta X_1} = \frac{a_{i2} - x_{i2}}{a_{i1} - x_{i1}} = \frac{1}{[(C_1{}^2)^{1/2}]^{1/p-1}}.$$

If the existing facility i is not on the linear facility, $x_2 = C_o + C_1x_1$, and $C_1 \neq 0$; then $x_{i1} = a_{i1}$ if and only if p equals the limit as p approaches 1 from the negative direction.

If the existing facility i is not on the linear facility, $x_2 = C_o + C_1x_1$, then the only way for $x_{i1} = a_{i1}$

is if the slope of the line connecting the two points is vertical (i.e., $\Delta X_2/\Delta X_1$ approaches infinity). Since the only way the slope can be equal to infinity is when $(1-p) = 0^-$ (i.e., the limit of (1-p) as p approaches 1 from the negative direction), then the limit of the slope as p approaches 1 from the negative direction is as follows:

$$\lim_{p\to 1^-}\left(\frac{-1}{[(C_1{}^2)^{1/2}]^{\frac{1}{p-1}}}\right)$$

$$= \lim_{p\to 1^-}\left(-[(C_1{}^2)^{1/2}]^{\frac{1/p}{1/p\ -\ p/p}}\right) = -\infty.$$

<u>Property 2.8</u>

If the existing facility i is not on the linear facility, $x_2 = C_o + C_1x_1$, and $C_1 \neq 0$; then $x_{i2} = a_{i2}$ if and only if p equals the limit as p approaches 1 from the positive direction.

<u>Proof</u>. If the existing facility i is not on the linear facility and $C_1 \neq 0$, then the only way $x_{i2} = a_{i2}$ is when the slope of the line connecting the two points is horizontal (i.e., $\Delta X_2/\Delta X_1 = 0$). Taking the limit as p approaches 1 from the positive direction, the following formula results:

$$\lim_{p \to 1^+} \left(- \frac{1}{[(C_1{}^2)^{1/2}]^{\frac{1}{p-1}}} \right)$$

$$= \lim_{p \to 1^+} \left(- \frac{1}{[(C_1{}^2)^{1/2}]^{\frac{1/p}{p/p - 1/p}}} \right) = 0.$$

Thus, the property is proven.

As can now be seen using Property 2.7, Equation (2.11) is defined for all values of p and q with the exception of p = 1. Likewise, as can be seen using Property 2.8, Equation (2.12) is defined for all values of p and q except p = 1. The first derivatives are undefined at p = 1 because the Lagrangian function is discontinuous at p = 1. The point (x_{i1}, x_{i2}) that minimizes the modified p distance from the existing facilities to the linear facility for p = 1 is as follows:

Limit as p approaches 1 from the negative direction:

$$x_{i1} = a_{i1},$$

$$x_{i2} = C_o + C_1 a_{i1}.$$

Limit as p approaches 1 from the positive direction:

$$x_{i1} = C_1(a_{i1} - C_o),$$

$$x_{i2} = a_{i2}.$$

Thus, the derivation of the point (x_{i1}, x_{i2}) is valid for all values of p and q, and the point (x_{i1}, x_{i2})

minimizes the p-norm distance taken to the q power with $q \geq 0$ and $p \geq 1$ or when $q < 0$ and $p < 1$.

Interpretation of x_{i1}, x_{i2}

The results of Property 2.7 indicate a physical interpretation for p within the modified distance function ℓ^q_{pi}. Working only with the case where $C_1 > 0$ (even though the conclusions are generalizable to $C_1 < 0$), the slope of the line connecting the existing facility to the linear facility is equal to $-1/[(C_1{}^2)^{1/2}]^{1/(p-1)}$. That the slope of this line is either vertical or horizontal as p approaches 1 from the negative or positive direction has been shown. That the slope of this same line is equal to -1 as p approaches infinity can also be shown. For any other value of p the slope of the line minimizing the modified p-norm distance is a function of the relationship between the values of p and C_1. If $p = 2$, for example, the slope is equal to $-1/[(C_1{}^2)^{1/2}]$ which is the perpendicular distance when $C_1 > 0$. As p gets larger, the absolute value of the slope of the line connecting (a_{i1}, a_{i2}) to the linear facility for a given value of C_1 gets larger.

Since the p-norm distance taken to the q power is minimized in the generalized, bivariate, linear location problem, the value of p determines which distance will be minimized. When p approaches 1 (i.e., the rectalinear case), the minimized distance is in the direction of only one of the variables. To use regression terminology, minimizing the error in only one direction assumes that one variable is fixed and that the other variable is random and accountable for the total error (i.e., independent and dependent variables). When p is not equal to 1, the error is divided between the two variables according to $-1/[(C_1^{\,2})^{1/2}]^{1/p-1}$, and the terms "independent" and "dependent" variables are no longer appropriate. Thus, p is the parameter within the generalized, bivariate, linear location function that determines the relative weight between the two variables with respect to the error term.

CHAPTER III

WEBER REFORMULATION: MODIFIED DISTANCE FUNCTION

As previously stated, locating a linear facility is a two-step process involving determination of (1) the point (x_{i1}, x_{i2}) on the linear facility "closest" to the existing facility i, and (2) estimates for the linear parameters C_o and C_1. In Chapter II a method of determining the point (x_{i1}, x_{i2}) on a line, $x_2 = C_o + C_1 x_1$, that minimizes the modified p-norm distance to the existing facility i located at the point (a_{i1}, a_{i2}) was illustrated.

Before estimates for the linear parameters C_o and C_1 can be determined, more detailed consideration must be given to the Weber formulation of the generalized, bivariate, linear location problem. In this chapter, therefore, the modified distance function $\ell^q_{pi}(C_o, C_1)$ associated with the p-norm distance taken to the q power from the existing facility i to the point (x_{i1}, x_{i2}), on the linear facility is developed. The convexity or lack of convexity of $\ell^q_{pi}(C_o, C_1)$ is also investigated. Finally, the Weber formulation of the linear location problem is presented.

Formulation of ℓ^q_{pi} (C_o, C_1)

The values of x_{i1} and x_{i2} developed in Chapter II can now be used to formulate the modified distance formula $\ell^q_{pi}(C_o, C_1)$. Inserting these values into Equation (2.1), the following two formulations corresponding to the pairs of values for x_{i1} and x_{i2} result:

$$\ell^q_{pi}(C_o, C_1) = \frac{[\{1 - [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{q/2}}{[\{1 - [C_1{}^2]^{p/(p-1)}\}^2]^{q/2}}$$

$$* \ \{1 + [(C_1{}^2)^{1/2}]^{p/(p-1)}\}^{q/p}$$

$$* \ [(a_{i2} - C_o - C_1 a_{i1})^2]^{q/2}, \tag{3.1a}$$

$$\frac{[\{1 + [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{p/2}}{[\{1 - [C_1{}^2]^{p/(p-1)}\}^2]^{p/2}}$$

$$* \ \{1 + [(C_1{}^2)^{1/2}]^{p/(p-1)}\}^{q/p}$$

$$* \ [(a_{i2} - C_o - C_1 a_{i1})^2]^{q/2}. \tag{3.1b}$$

The solution in Equation (3.1a) resulted from using the first solutions for x_{i1} and x_{i2} (i.e., Equations (2.15a) and (2.16a)), and the second solution in Equation (3.1b) resulted from use of Equations (2.15b) and (2.16b). The only difference between Equations (3.1a) and (3.1b) is found in the following terms:

$$[\{1 - [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{q/2)}, \tag{3.2a}$$

$$[\{1 + [C_1][(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{q/2}. \tag{3.2b}$$

Remembering that the first pair of solutions is valid only when $C_1 \geq 0$ and that the second pair of solutions is valid only when $C_1 \leq 0$, the following operations can be performed:

$$[\{1 - |C_1|[(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{q/2}, \tag{3.3a}$$

$$[\{1 - |C_1|[(C_1{}^2)^{1/2}]^{1/(p-1)}\}^2]^{q/2}. \tag{3.3b}$$

Thus, the two terms are identical, and the following single formulation for $\ell^q_{pi}(C_o, C_1)$ results:

$$\ell^q_{pi}(C_o, C_1) = \frac{[(a_{i2} - C_o - C_1 a_{i1})^2]^{q/2}}{[1 + [(C_1{}^2)^{1/2}]^{p/(p-1)}]^{q(p-1)/p}}. \tag{3.4}$$

A simplified definitional formula for ℓ^q_{pi} is as follows:

$$\ell^q_{pi}(C_o, C_1) = \frac{|a_{i2} - C_o - C_1 a_{i1}|^q}{|1 + |C_1|^{p/(p-1)}|^{q(p-1)/p}} \tag{3.5}$$

Convexity Properties of L^q_{pi}

To investigate the convexity of the modified distance function $\ell^q_{pi}(C_o, C_1)$ the first- and second-order partial derivatives of $\ell^q_{pi}(C_o, C_1)$ with respect to C_o and C_1 must be taken (Hillier & Lieberman, 1980). Using either the calculation Formula (3.4) or the Definitional Formula (3.5), similar difficulties to those encountered in Equation (2.5) are encountered with respect to undefined rth partial derivatives if $q \leq r$ and $a_{i2} = C_o + C_1 a_{i1}$. To remove these problems, a smoothing constant m similar to the concept used by Morris and Verdini (1979) for the point location problem can be employed. Using the hyperbolic smoothing constant m, the following equation is an approximation for $\ell^q_{pi}(C_o, C_1)$:

$$L^q_{pi}(C_o, C_1) = \frac{[V_i^2 + m]^{q/2}}{G^{q(p-1)/p}} \tag{3.6}$$

where

$V_i = a_{i2} - C_o - C_1 a_{i1}$ for $i = 1, \ldots, n$,

$G = 1 + [(C_1^2)^{1/2}]^{[p/(p-1)]}$,

$p \geq 1$,

$q \geq 0$, and

$m \geq 0$.

One definition and several theorems from Protter and Morrey (1964) are helpful with respect to the proofs of properties within this section.

Definition 3.1

A function f is said to be continuous at (b, d) if and only if:

(a) f is defined at (b, d), and

(b) $\lim_{(x, y) \to (b, d)} f(x, y) = f(b, d)$.

Theorem 3.1

If b is a constant and f(x) = b for all values of x, then for any number d

$$\lim_{x \to d} f(x) = b.$$

Theorem 3.2

If m and b are constants and f(x) = mx + b for all values of x, then for any number d

$$\lim_{x \to d} f(x) = md + b.$$

Theorem 3.3

If f and q are two functions with

$$\lim_{x \to d} f(x) = M_1, \lim_{x \to d} q(X) = M_2,$$

then

$$\lim_{x \to d} \frac{f(x)}{g(x)} = \frac{M_1}{M_2} \text{ where } M_2 \neq 0$$

and

$$\lim_{x \to d} [f(x) * g(x)] = M_1 * M_2.$$

Theorem 3.4

If f and g are two functions with

$$\lim_{x \to d} f(x) = M_1 \qquad \lim_{x \to d} g(x) = M_2,$$

then

$$\lim_{x \to d} [f(x) + g(x)] = M_1 + M_2.$$

Theorem 3.5

If n is any positive integer, and if f is such that

$$\lim_{x \to d} f(x) = M;$$

then

$$\lim_{x \to d} [f(x)]^{1/n} = M^{1/n}.$$

Property 3.1

The limit of $L^q_{pi}(C_o, C_1)$ as m approaches zero is

$$\ell^q_{pi}(C_o, C_1).$$

Proof. Using theorem 3.1, the following can be stated:

$$\lim_{m \to 0} V_i^{\,2} = V_i^{\,2} \text{ and } \lim_{m \to 0} G^{q(p-1)/p} = G^{q(p-1)/p} = L_1^*.$$

Using theorem 3.2, the following can be stated:

$$\lim_{m \to 0} [V_i^2 + m] = V_i^2 = L_2^*.$$

Finally, using theorems 3.3 and 3.5 and the knowledge that q/2 may be represented by a rational number without severely limiting the domain of q, then

$$\lim_{m \to 0} L_{pi}^q(C_o, C_1) = \lim_{m \to 0} \{\frac{[L_2^*]^{q/2}}{L_1^*}\}$$

$$= \frac{[L_2^*]^{q/2}}{L_1^*} = \ell_{pi}^q(C_o, C_1). \quad \text{(Q.E.D.)}$$

The first- and second-order partial derivatives of $L_{pi}^q(C_o, C_1)$ with respect to C_o and C_1 (i.e., Equation (3.6)) are as follows:

$$\frac{\partial L_{pi}^q(C_o, C_1)}{\partial C_o} = [\frac{-q}{G^{q(p-1)/p}}][V_i^2 + m]^{(q/2)-1} V_i, \quad (3.7)$$

$$\frac{\partial^2 L_{pi}^q(C_o, C_1)}{\partial C_o^2} = [\frac{q}{G^{q(p-1)/p}}][V_i^2 + m]^{(q/2)-2}$$

$$* [V_i^2(q-1) + m], \quad (3.8)$$

$$\frac{\partial L^q_{pi}(C_o, C_1)}{\partial C_1} = [\frac{-q}{G^{[q(p-1)/p]+1}}][V_i^2 + m]^{(q/2)-1}$$

$$* \{GV_i a_{i1} + C_1[(C_1^2)^{1/2}]^{(2-p)/(p-1)}$$

$$* (V_i^2 + m)]\}, \qquad (3.9)$$

$$\frac{\partial^2 L^q_{pi}(C_o, C_1)}{\partial C_1^2} = \{\frac{q[V_i^2 + m]^{(q/2)-2}}{G^{[q(p-1)/p] + 2}}\}\{G^2 a_{i1}^2[V_i^2(q-1) + m]$$

$$+ \quad 2qCa_{i1}V_iG[(C_1^2)^{1/2}]^{(2-p)/(p-1)}[V_i^2 + m]$$

$$+ \quad \frac{[(C_1^2)^{1/2}]^{(2-p)/(p-1)}[V_i^2 + m]^2}{(p-1)}$$

$$* \quad [(q + 1)(p - 1)[(C_1^2)^{1/2}]^{p/(p-1)} - 1\}, \qquad (3.10)$$

$$\frac{\partial^2 L^q_{pi}(C_o, C_1)}{\partial C_o \partial C_1} = \{\frac{q[V_i^2 + m]^{(q/2)-2}}{G^{[q/(p-1)/p]+1}}\}\{Ga_{i1}[V_i^2(q-1) + m]$$

$$+ qC_1[(C_1^2)^{1/2}]^{2-p/p-1}V_i[V_i^2 + m]\}. \qquad (3.11)$$

<u>Property 3.2</u>

$L^q_{pi}(C_o, C_1)$ is continuous for $q \geq 0$ and $p > 1$.

<u>Proof</u>. Let $f_i(C_o, C_1) = L^q_{pi}(C_o, C_1)$, and consider $f_i(C_o, C_1)$ at any point $c_o \in C_o$ and $c_1 \in C_1$ where c_o and c_1 are real numbers. By inspection, requirement (a) of

Definition 3.1 is satisfied since $f_i(c_o, c_1) = [V^2 + m]^{q/2}/G^{q(p-1)/p}$, $i = 1, \ldots, n$ are defined for all $c_o \varepsilon C_o$ and $c_1 \varepsilon C_1$ and $q \geq 0$ and $p > 1$. Requirement (b) is satisfied because

$$\lim_{(b, d) \to (c_o, c_1)} f_i(b, d) = f_i(c_o, c_1)$$

for all $i = 1, \ldots, n$ and $c_o \varepsilon C_o$ and $c_1 \varepsilon C_1$. The proof for the preceding equation is similar to the proof in Property 3.1. Therefore, the function is continuous for $q \geq 0$ and $p > 1$.

The case of $p = 1$ is not covered by the preceding and subsequent properties because $f_i(C_o, C_1)$ is undefined at $p = 1$. The situation where $p = 1$ is addressed separately with properties of the Weber formulation.

Property 3.3

The first- and second-order partial derivatives of $L^q_{pi}(C_o, C_1)$ with respect to C_o and C_1 are continuous for $q \geq 0$ and $p > 1$.

Proof. Let $f'_{io}(C_o, C_1) = \partial L^q_{pi}(C_o, C_1)/\partial C_o$, $f'_{i1}(C_o, C_1) = \partial L^q_{pi}(C_o, C_1)/\partial C_1$, $f''_{io} = \partial^2 L^q_{pi}(C_o, C_1)/\partial C_o{}^2$, $f''_{i1}(C_o, C_1) = \partial^2 L^q_{pi}(C_o, C_1)/\partial C_1{}^2$, and $f''_{i1o}(C_o, C_1) = \partial^2 L^q_{pi}(C_o, C_1)/\partial C_1 C_o$; and consider $f'_{io}(C_o, C_1)$, $f'_{i1}(C_o, C_1)$, $f''_{io}(C_o, C_1)$, $f''_{i1}(C_o, C_1)$ and $f''_{i1o}(C_o, C_1)$ at any point $c_o \varepsilon C_o$ and $c_1 \varepsilon C_1$ where c_o and c_1 are real numbers. Again, by inspection $f'_{io}(c_o, c_1)$, $f'_{i1}(c_o, c_1)$, $f''_{io}(c_o, c_1)$, $f''_{i1}(c_o, c_1)$, and $f''_{i1o}(c_o, c_1)$ satisfy requirement (a) of Definition 3.1 since they are defined for all $c_o \varepsilon C_o$ and $c_1 \varepsilon C_1$, $i = 1, \ldots, n$ and $q \geq 0$ and $p > 1$. Requirement (b) is satisfied because

$$\lim_{(b,\ d)\to(c_o,\ c_1)} f'_{io}(b, d) = \frac{\partial L^q_{pi}(c_o, c_1)}{\partial C_o},$$

$$\lim_{(b,\ d)\to(c_o,\ c_1)} f'_{i1}(b, d) = \frac{\partial L^q_{pi}(c_o, c_1)}{\partial C_1},$$

$$\lim_{(b,\ d)\to(c_o,\ c_1)} f''_{io}(b, d) = \frac{\partial^2 L^q_{pi}(c_o, c_1)}{\partial C_1{}^2},$$

$$\lim_{(b,\ d)\to(c_o,\ c_1)} f''_{i1}(b, d) = \frac{\partial^2 L^q_{pi}(c_o, c_1)}{\partial C_1{}^2},$$

$$\lim_{(b,\ d)\to(c_o,\ c_1)} f''_{i1o}(b, d) = \frac{\partial^2 L^q_{pi}(c_o, c_1)}{\partial C_1 \partial C_o},$$

for all $i = 1, \ldots, n$ and $c_o \ \varepsilon \ C_o$ and $c_1 \ \varepsilon \ C_1$. The proofs for the preceding equations are similar to the proof in Property 3.1. Therefore, the property is proven.

Property 3.4

$L^q_{pi}(C_o, C_1)$ is strictly convex with respect to C_o for $q \geq 1$ and $m > 0$ and strictly concave for $0 < q < 1$ and the limit as m approaches but does not equal zero.

Proof. Since $L^q_{pi}(C_o, C_1)$ has continuous second derivatives for all points $c_o \varepsilon C_o$ where c_o is a real number, it is sufficient to show that the second-order partial derivative with respect to C_o is strictly positive for $q \geq 1$ and negative for $0 < q < 1$ where the limit of m approaches but does not equal zero. From equation (3.8) the second-order partial derivative of $L^q_{pi}(C_o, C_1)$ with respect to C_o is given by

$$\frac{\partial^2 L^q_{pi}(C_o, C_1)}{\partial C_o{}^2} = \frac{q}{G^{q(p-1)/p}}[V_i{}^2 + m]^{(q/2)-2}$$

$$* \ [V_i{}^2(q-1) + m].$$

Since G and m are always positive, $\partial^2 L^q_{pi}(C_o, C_1)/\partial C_o{}^2$ is strictly positive for $q \geq 1$. If $0 < q < 1$, $\partial^2 L^q_{pi}(C_o, C_1)/\partial C_o{}^2$ is strictly negative as m approaches but does not equal zero.

Properties of the Weber Formulation

Substituting Equation (3.5) into the Weber formulation produces the following generalized, bivariate, linear location problem:

$$C\ell^q_p(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + |C_1|^{p/(p-1)}]^{q(p-1)/p}}. \qquad (3.12)$$

Again using the hyperbolic approximation function to ensure defined derivatives and to eliminate computational difficulties associated with the absolute value functions, the following approximation of $C\ell_p^q(C_o, C_1)$ results from substituting Equation (3.6) into the Weber formulation:

$$CL_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{q(p-1)/p}}. \tag{3.13}$$

Earlier the statement was made that the previous properties in this chapter were applicable only when $p > 1$. The following properties relieve this restriction for $p = 1$.

Property 3.5

The limit of $C\ell_p^2(C_o, C_1)$ as p approaches 1 from the negative direction is the regression of x_2 on x_1 taken to the q power.

Proof. The limit of $G^{q(p-1)/p}$ as p approaches 1 from the negative direction is as follows:

$$\lim_{p \to 1^-} [1 + |C_1|^{p/(p-1)}] = [1 + 0]^0 = 1. \tag{3.14}$$

Thus, the Weber formulation, when p approaches 1 from the negative direction, is as follows:

$$C\ell_p^q(C_o, C_1) = \sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q. \qquad (3.15)$$

When $W_i = 1$ for $i = 1, \ldots, n$ and $q = 2$, the generalized, bivariate, linear location problem reduces to the least squares regression of x_2 on x_1.

Property 3.6

The limit of $C\ell_p^q(C_o, C_1)$ as p approaches 1 from the positive direction is the regression of x_1 and x_2 taken to the q power.

Proof. From Equation (3.15) $C\ell_p^q(C_o, C_1)$ is equal to the following equation:

$$C\ell_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i \mid a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + |C_1|^{p/(p-1)}]^{q(p-1)/p}} .$$

Dividing the numerator and denominator by $1/|C_1|^q$, $C\ell_p^q(C_o, C_1)$ becomes

$$C\ell_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |(a_{i2} - C_o - C_1 a_{i1})/C_1|^q}{[1/|C_1|^{p/(p-1)} + 1]^{q(p-1)/p}}.$$

Taking the limit of $C\ell_p^q(C_o, C_1)$ as p approaches 1 from the positive direction, the following result is obtained:

$$\lim_{p \to 1^+} [C\ell_p^q(C_o, C_1)] = \sum_{i=1}^{n} w_i |(a_{i2} - C_o - C_1 a_{i1})/C_1|^q. \tag{3.16a}$$

Rearranging the linear constraint, $x_2 = C_o + C_1 x_1$, to $x_1 = (x_2 - C_o)/C_1$ and defining $C_1' = 1/C_1$ and $C_o' = -C_o/C_1$, the limit of $C\ell_p^q(C_o, C_1)$ as p approaches one from the positive direction may be stated as

$$C\ell_p^q(C_o', C_1') = \sum_{i=1}^{n} w_i |a_{i1} - C_o' - C_1' a_{i2}|^q \tag{3.16b}$$

where

$$C_o' = \frac{-C_o}{C_1} \text{ and } C_1' = \frac{1}{C_1} .$$

Equation (3.16b) is just the statement of the objective function for regression x_1 on x_2. As can be seen, regression of x_1 on x_2 may be solved indirectly as a special case of the generalized, bivariate, linear location problem. The preceding result is supported by Property 2.7 where the statement was made that the slope of the line connecting the existing facility i to the point on the linear facility that minimizes the p-norm distance taken to the q power is horizontal (i.e., equal to zero) when the limit is taken as p approaches 1 from the positive direction.

Property 3.7

$CL_p^q(C_o, C_1)$ is strictly convex with respect to C_o for $q \geq 1$ and $m > 0$ and strictly concave for $0 < q < 1$ and the limit as m approaches but does not equal zero.

Proof. This property follows since $CL_p^q(C_o, C_1)$ is the sum of strictly convex terms when $q \geq 1$ and $m > 0$ and strictly concave terms when $0 < q < 1$ and the limit as m approaches but does not equal zero.

No general statement can be made at this point of the analysis concerning the overall convexity or lack of convexity of $CL_p^q(C_o, C_1)$ since this property changes with different values of p and q. By Property 3.7 $CL_p^q(C_o, C_1)$ is not convex with respect to C_o for values of $0 < q < 1$. This is a sufficient condition for stating that $CL_p^q(C_o, C_1)$ is not convex within the range $0 < q < 1$ (Hillier & Lieberman, 1980). Yet, by Properties 3.5 and 3.6 $CL_p^q(C_o, C_1)$ reduces to simple regression when $q = 2$ and the limit is taken as p approaches 1 in either the plus or minus direction and simple regression is strictly convex (Neter & Wasserman, 1974). This inconsistency requires that convexity be investigated for each individual combination of p and q.

Summary

Detailed in this chapter is the Weber formulation of the generalized, bivariate, linear location problem. The problems associated with undefined derivatives require the addition of a hyperbolic smoothing constant. The modified Weber formulation of the bivariate, linear location problem is not necessarily concave or convex.

CHAPTER IV

SOLUTIONS TO THE GENERALIZED, BIVARIATE, LINEAR LOCATION PROBLEM

The complex structure of the Definitional Formula, Equation (3.12), and the Weber Reformulation Equation (3.13), along with the associated nonconvexity prohibits a universal analytical technique for solving the generalized, bivariate, linear location problem. Yet, the situation is not hopeless for those requiring analytical solutions. Just as illustrated in Chapter III, linear regression (i.e., $q = 2$ and p approaches 1 from either the positive or the negative direction) is one special case of the generalized, bivariate, linear location problem. Other special combinations of p's and q's likewise, reduce to common forms of analysis that are currently employed.

In this chapter the relationship between the generalized, bivariate, linear location problem and current forms of analysis is illustrated. In addition, certain unique combinations of p's and q's that represent models not presently employed are solved by various analytical or numerical techniques. Finally, a heuristic technique is presented for a general solution of the generalized, bivariate, linear location problem.

The Limit as p Approaches 1 Family

If, in a manner similar to that used in the proof of Property 3.5, the limit is taken as p approaches 1 from the negative direction, the definitional formulation of the generalized, bivariate, linear location problem Equation (3.12) becomes the following formula:

$$C\ell_{1^-}^{q}(C_o, C_1) = \sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q. \qquad (4.1)$$

The distance being considered in this family of models is the vertical distance from the existing facilities to the linear facility to be located taken to the power q.

As shown in Property 3.5 when $q = 2$, Equation (4.1) becomes

$$C\ell_{1^-}^{2}(C_o, C_1) = \sum_{i=1}^{n} W_i (a_{i2} - C_o - C_1 a_{i1})^2 \qquad (4.2)$$

which is simply the regression of x_2 on x_1. The solution for this well-known model is found by using the classical unconstrained optimization technique of taking the partial derivatives of Equation (4.2) with respect to C_o and C_1 and setting each result equal to zero. The resulting so-called "normal equations" are then solved simultaneously (Neter & Wasserman, 1974).

If q = 1 in Equation (4.1), the solution technique for the generalized, bivariate, linear location problem is not as straightforward. The following result, after the substitution, prohibits the use of the classical optimization technique:

$$C\ell_1^1\text{-}(C_o, C_1) = \sum_{i=1}^{n} W_i|a_{i2} - C_o - C_1a_{i1}|. \qquad (4.3)$$

One of the first solution techniques used to solve Equation (4.3) was proposed by Charnes, Cooper, and Ferguson (1955). They demonstrated that the following linear program produced an optimal, although not necessarily an unique, solution (Sposito & Smith, 1976).

Let d_i^+ and d_i^- be the vertical deviations "above" and "below" the linear facility to be located for the ith existing facility. Thus, for any existing facility i

$$C_1a_{i1} - C_o + d_i^+ - d_i^- = a_{i2}. \qquad (4.4)$$

The simplex formulation of the problem takes on the following structure:

$$\text{Minimize} \sum_{i=1}^{n} (W_id_i^+ + W_id_i^-)$$

subject to

$$C_1 a_{i1} + C_o + d_i^+ - d_i^- = a_{i2} \quad \forall_i,$$

$$d_i^+, d_i^- \geq 0, \text{ and}$$

$$C_1, C_o \text{ are unrestricted.} \tag{4.5}$$

With modifications for unrestricted variables, any simplex program can be used to solve this linear programming formulation.

Linear programming can be used to solve another member of this family--specifically, when q approaches infinity (Chebyshev criterion). When q approaches infinity, the problem becomes the location of the linear facility such that the maximum vertical distance from the existing facility to the linear facility is minimized. Wagner (1959) demonstrates the formulation which follows. Again, let d represent the deviations from the existing facility to the linear location line to be located. In this case, however, let d be common to each of the following inequalities:

$$-d \leq C_1 a_{i1} + C_o - a_{i2} \leq +d \quad \forall_i. \tag{4.6}$$

Therefore, the linear program becomes

Minimize d

subject to

$$C_1 a_{i1} + C_o - d \leq a_{i2} \qquad \forall_i,$$

$$C_1 a_{i1} + C_o + d \geq a_{i2} \qquad \forall_i,$$

$$d \geq 0, \text{ and}$$

$$C_1 \text{ and } C_o \text{ are unrestricted.} \qquad (4.7)$$

Again, any simplex program can be used to solve the preceding formulation (i.e., with slight modification for unrestricted variables depending upon the program).

Forsythe (1972) extended the range of analytical solutions for this family for values of $1 \leq q \leq 2$ by the use of gradient search techniques, and Morris and Norback (1980) extended the range for values of q between 2 and infinity by the use of convex programming and Kuhn-Tucker conditions. The applicability of these techniques results from the property which follows.

Property 4.1

$CL_p^q(C_o, C_1)$ is convex with resptct to both C_o and C_1 when the limit is taken as p approaches 1 from either the positive or the negative direction and $q \geq 1$.

Proof. The Weber formulation with the hyperbolic smoothing constant for the generalized, bivariate, linear location problem when the limit is taken as p approaches 1 from the negative direction is as follows:

$$CL_{1^-}^{q}(C_o, C_1) = \sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}. \tag{4.8}$$

The first partial derivatives of $CL_{1^-}^{q}(C_o, C_1)$ with respect to C_o and C_1 are as follows:

$$\frac{\partial CL_{1^-}^{q}(C_o, C_1)}{\partial C_o} = -q\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-1} * (a_{i2} - C_o - C_1 a_{i1}), \tag{4.9}$$

$$\frac{\partial CL_{1^-}^{q}(C_o, C_1)}{\partial C_1} = -q\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-1} * (a_{i2} - C_o - C_1 a_{i1})a_{i1}. \tag{4.10}$$

The second partial derivatives of $CL_{1^-}^{q}(C_o, C_1)$ with respect to C_o and C_1 are as follows:

$$\frac{\partial^2 CL_{1^-}^{q}(C_o, C_1)}{\partial C_o{}^2} = q\{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-2} * [(a_{i2} - C_o - C_1 a_{i1})^2(q - 1) + m]\}, \tag{4.11}$$

$$\frac{\partial^2 CL^q_{1-}(C_o, C_1)}{\partial C_1^{\,2}} = q\{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1a_{i1})^2 + m]^{(q/2)-2}$$

$$* [(a_{i2} - C_o - C_1a_{i1})^2(q-1) + m]a_{i1}\}, \quad (4.12)$$

$$\frac{\partial^2 CL^q_{1-}(C_o, C_1)}{\partial C_o \partial C_1} = q\{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1a_{i1})^2 + m]^{(q/2)-2}$$

$$* [(a_{i2} - C_o - C_1a_{i1})^2(q-1) + m]a_{i1}\}. \quad (4.13)$$

The determinant of the Hessian, D, for the family of functions where the limit is taken as p approaches 1 from the negative direction, Equation (4.1), is as follows:

$$D = \left(\frac{\partial^2 CL^q_{1-}(C_o, C_1)}{\partial C_o^{\,2}}\right)\left(\frac{\partial^2 CL^q_{1-}(C_o, C_1)}{\partial C_1^{\,2}}\right) - \left(\frac{\partial^2 CL^p_{1-}(C_o, C_1)}{\partial C_o C_1}\right)^2,$$

$$= q^2[\{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1a_{i1})^2 + m]^{(q/2)-2}$$

$$* [(a_{i2} - C_o - C_1a_{i1})^2(q - 1) + m]\}$$

$$* \{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1a_{i1})^2 + m]^{(q/2)-2}$$

$$* [(a_{i2} - C_o - C_1a_{i1})^2(q - 1) + m]a_{i1}^2\}$$

$$- \{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1a_{i1})^2 + m]^{(q/2)-2}$$

$$* [(a_{i2} - C_o - C_1a_{i1})^2(q - 1) + m]a_{i1}\}^2]. \quad (4.14)$$

If this determinant (4.14) is positive, it is a sufficient condition for convexity since both Equations (4.11) and (4.12) are positive when $q \geq 1$. The sign of Equation (4.14) is difficult to evaluate, however, because of the summations. An indirect method of determining the convexity properties of $CL^q_{1-}(C_o, C_1)$ is examination of the function on a term-by-term basis (i.e., $L^q_{1-i}(C_o, C_1)$).

Given that

$$L^q_{1-i}(C_o, C_1) = [(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}, \quad (4.15)$$

the relevant partial derivatives with respect to C_o and C_1 are as follows:

$$\frac{\partial L^q_{1-i}(C_o, C_1)}{\partial C_o} = -q[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)^{-1}} * (a_{i2} - C_o - C_1 a_{i1}), \quad (4.16)$$

$$\frac{\partial L^q_{1-i}(C_o, C_1)}{\partial C_1} = -q[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-1} * (a_{i2} - C_o - C_1 a_{i1})a_{i1}, \quad (4.17)$$

$$\frac{\partial^2 L^q_{1-i}(C_o, C_1)}{\partial C_o^{\,2}} = q[(a_{i1} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-2} * [(a_{i2} - C_o - C_1 a_{i1})^2(q - 1) + m], \quad (4.18)$$

$$\frac{\partial^2 L_{1-i}^{q}(C_o, C_1)}{\partial C_1^{\,2}} = q[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-2}$$

$$* \; [(a_{i2} - C_o - C_1 a_{i1})^2 (q-1) + m] a_{i1}^{\,2}, \quad (4.19)$$

$$\frac{\partial^2 L_{1-i}^{q}(C_o, C_1)}{\partial C_o \partial C_1} = q[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-2}$$

$$* \; [(a_{i2} - C_o - C_1 a_{i1})^2 (q-1) + m] a_{i1}. \quad (4.20)$$

The determinant of the Hessian, D, for $L_{1-i}^{q}(C_o, C_1)$ is given by the following formula:

$$D = \left(\frac{\partial^2 L_{1-i}^{q}(C_o, C_1)}{\partial C_o^{\,2}}\right)\left(\frac{\partial^2 L_{1-i}^{q}(C_o, C_1)}{\partial C_1^{\,2}}\right) - \left(\frac{\partial^2 L_{1-i}^{q}(C_o, C_1)}{\partial C_o \partial C_1}\right)^2,$$

$$= q^2[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{(q/2)-2}$$

$$* \; [(a_{i2} - C_o - C_1 a_{i1})^2 (q - 1) + m][a_{i1}^{\,2} - a_{i1}^{\,2}],$$

$$= 0. \quad (4.21)$$

Since Equations (4.18) and (4.19) are positive when $q > 1$ and each element of this determinant is equal to zero, $L_{1-i}^{q}(C_o, C_1)$ is convex. This property implies that $CL_{1-}^{q}(C_o, C_1)$ is convex since it is the sum of convex terms, and the property is proven.

For $0 < q < 1$ Equations (4.11) and (4.12) are positive indicating $CL_{1-}^{q}(C_o, C_1)$ is concave. Therefore, gradient search techniques and convex programming are not conducive to determining optimality. Barrodale and Roberts (1970) proved that the values of C_o and C_1 which optimize Equation (4.1) when $0 < q < 1$ must pass through at least two points (a_{i1}, a_{i2} and a_{k1}, a_{k2} where $i \neq k$). Thus, a simplistic solution is the determination of each possible line that passes through two data points and determination of the value of $C\ell_{1}^{q}$-(C_o, C_1) for each line. (See Barrodale & Roberts, 1970, for a proof of the above procedure. A variation of the proof is presented later in this chapter.)

The p = 2 Family

The Weber reformulation of the p = 2 family of the generalized, bivariate, linear location problem is given by the following formula which is a special case of Equation (3.13):

$$CL_2^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}}{[1 + C_1{}^2]^{q/2}} . \tag{4.22}$$

The members of this family of the generalized, bivariate, linear location problem seek values of C_o and C_1 that minimize the perpendicular distance from the existing facilities to the linear location facility taken to the q power. Although the publications in this area have not been as plentiful as those in the p = 1 family, extensive research has been done for certain cases.

The q = 2 model is the most common application of the p = 2 family. More commonly known as factor analysis, this model determines values for C_o and C_1 that minimize the expression

$$CL_2^2(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i[a_{i2} - C_o - C_1a_{i1}]^2}{[1 + C_1^2]}. \qquad (4.23)$$

Classical optimization can be used to solve the preceding expression even though the typical method employed involves formulating the eigen-structure (Green, 1978). In the simple bivariate case, the following procedure is adequate.

1. Determine the first-order partial derivatives:

$$\frac{\partial CL_2^2(C_o, C_1)}{\partial C_o} = -2\sum_{i=1}^{n}\frac{W_i[a_{i2} - C_o - C_1a_{i1}]}{[1 + C_1^2]}. \qquad (2.24)$$

$$\frac{\partial CL_2^2(C_o, C_1)}{C_1} = -2\ \{\sum_{i=1}^{n} W_i[a_{i2} - C_o - C_1a_{i1}][1 + C_1^2]a_{i1}$$

$$+ C_1\sum_{i=1}^{n} W_i[a_{i2} - C_o - C_1a_{i1}]^2\}. \qquad (4.25)$$

2. Set Equations (4.24) and (4.25) equal to zero, and solve simultaneously:

$$C_1 = \frac{-A \pm (A^2 + 4B^2)^{1/2}}{2B} \tag{4.26}$$

where

$$A = \sum_{i=1}^{n} W_i [\sum_{i=1}^{n} W_i a_{i2}^2 - \sum_{i=1}^{n} W_i a_{i1}^2] + (\sum_{i=1}^{n} W_i a_{i1})^2 - (\sum_{i=1}^{n} W_i a_{i2})^2,$$

$$B = \sum_{i=1}^{n} W_i a_{i1} \sum_{i=1}^{n} W_i a_{i2} - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} W_i a_{i1} a_{i2}, \text{ and}$$

$$C_o = \frac{\sum_{i=1}^{n} W_i [a_{i2} - C_1 a_{i1}]}{\sum_{i=1}^{n} W_i}. \tag{4.27}$$

The preceding formulation will result in two solutions: maximizing Equation (4.23) and minimizing Equation (4.23).

Morris and Norback (1980) extended the work of Wesolowsky's numerical technique into an analytic solution for Equation (4.22) when $0 < q \leq 1$. Using a technique similar to that of Barrodale and Roberts (1970) for the vertical distance, Morris and Norback proved that a line minimizing Equation (4.22) must also pass through at least two existing facilities. Thus, the solution technique for $0 < q \leq 1$ and $p = 2$ is identical to the technique used to solve the limit as p approaches 1 from the negative direction family.

Morris and Norback pointed out, in the same article, that the techniques used to solve the vertical distance problem for $q > 1$ cannot be used for the $p = 2$ family since Equation (4.22) becomes nonconvex and nonconcave when $q > 1$.

The $q = 2$ Family

The $q = 2$ family of the generalized, bivariate, linear location problem is significant for two important reasons: (1) linear regression (p approaches 1^- and $q = 2$) and factor analysis ($p = 2$, $q = 2$) are the most widely used applications of linear models today, and (2) the mean which is the basis of the $q = 2$ model allows for simplified solution techniques that provide analytical results regardless of the p value. Since parametric statistics are based upon the mean, the $q = 2$ model has a very strong theoretical attractiveness even though it is overly sensitive to outliers. The simplified solution techniques for the $q = 2$ model result from the property which follows.

Property 4.2

Given the function

$$CL_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i[(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}}{[1 + [(C_1{}^2)^{1/2}]^{p/p-1}]^{q(p-1)/p}}, \qquad (4.28)$$

the values of C_o^* and C_1^* that minimize $CL_p^2(C_o, C_1)$ result in a line that passes through the point $(\bar{a}_i, \bar{a}_2)$ where

$$\bar{a}_1 = \frac{\sum_{i=1}^{n} W_i a_{i1}}{\sum_{i=1}^{n} W_i} \text{ and } \bar{a}_2 = \frac{\sum_{i=1}^{n} W_i a_{i2}}{\sum_{i=1}^{n} W_i} .$$

Proof. If $q = 2$, then Equation (4.28) reduces to:

$$CL_p^2(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i (a_{i2} - C_o - C_1 a_{i1})^2}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{2(p-1)/p}}. \qquad (4.29)$$

Let $a_{i1}^* = a_{i1} - \bar{a}_1$ and $a_{i2}^* = a_{i2} - \bar{a}_2$ be the mean corrected values for the existing facilities (a_{i1}, a_{i2}). Substituting the values a_{i1}^* and a_{i2}^* into Equation (4.29) and taking the first partial derivative with respect to C_o, the *following* result is obtained:

$$\frac{\partial L_p^2(C_o, C_1)}{\partial C_o} = \frac{-2\sum_{i=1}^{n} W_i(a_{i1}^* - C_o - C_1 a_{i1}^*)}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{2(p-1)/p}}. \quad (4.30)$$

Setting Equation (4.30) equal to zero and substituting $a_{i1}^* = a_{i1} - \overline{a}_1$ and $a_{i2}^* - \overline{a}_2$ results in the following equation:

$$\sum_{i=1}^{n} W_i(a_{i2} - \overline{a}_2 - C_o - C_1(a_{i1} - \overline{a}_1)) = 0,$$

$$\sum_{i=1}^{n} W_i a_{i2} - \sum_{i=1}^{n} W_i \overline{a}_2 - \sum_{i=1}^{n} W_i C_o - C_1 \sum_{i=1}^{n} W_i a_{i1}$$

$$+ \quad C_1 \sum_{i=1}^{n} W_i \, a_{i1} = 0,$$

and therefore, $$C_o = 0. \quad (4.31)$$

By mean correcting the data, the coordinates are translated such that the point $(\overline{a}_i, \overline{a}_2)$ is located at (0, 0). Since $C_o \equiv 0$ when the data is mean corrected regardless of the p value, the property is proven.

With the knowledge that $C_1 \equiv 0$ when $q = 2$ and the data is mean corrected, Equation (4.29) can be rewritten as follows:

$$CL_p^2(C_1) = \frac{\sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)^2}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{2(p-1)/p}}. \tag{4.32}$$

Taking the first derivative with respect to C_1 and setting it equal to zero produces the following results:

$$\frac{dCL_p^2(C_1)}{dC_1} = \frac{-2 \sum_{i=1}^{n} W_i}{[1 + [(C_1^2)^{1/2}]^{[2(p-1)/p]+1}}$$
$$* \{[1 + [(C_1^2)^{1/2(p/p-1)}](a_{i2}^* - C_1 a_{i1}^*)a_{i1}^*$$
$$+ C_1[(C_1^2)^{1/2}]^{(2-p)/(p-1)}(a_{i2}^* - C_1 a_{i1}^*)\}, \tag{4.33}$$

$$[(C_1^2)^{1/2}]^{(p-2)/(p-1)} \sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)a_{i1}^*$$
$$+ C_1^2 \sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)a_{i1}^*$$
$$+ C_1 \sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)^2$$
$$= 0, \tag{4.34}$$

$$[(C_1^2)^{1/2}]^{1/(p-1)} \{[\frac{\sum_{i=1}^{n} W_i[a_{i2}^* - C_1 a_{i1}^*]a_{i2}^*}{\sum_{i=1}^{n} W_i[a_{i2}^* - C_1 a_{i1}^*]a_{i1}^*}]^2\}^{1/2} = 1. \tag{4.35}$$

Let p be equal to a rational number u/v. Then Equation (4.35) can be reduced to:

$$C_1^{2v}[\sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)a_{i2}^*]^{2(u-v)}$$

$$- [\sum_{i=1}^{n} W_i(a_{i2}^* - C_1 a_{i1}^*)a_{i1}^*]^{2(u-v)} = 0. \qquad (4.36)$$

Given the assumption that p is a rational number (i.e., not an overly restrictive assumption), Equation (4.36) can be solved for any value of p. If the resulting polynomial is greater than order 4, a numerical technique such as the Birge-Vieta iterative method (McCalla, 1967) is required to determine all possible solutions for C_1. The value of C_1 that minimizes Equation (4.32) can then be found by straight substitution. Theoretically, the procedure outlined in this section is applicable for any values of p (i.e., as long as p can be considered a rational number). Certain values of p, however, result in polynomials of excessively large order (i.e., p = 1.51 produces a 301th order polynomial, while p = 1.50 produces only a 6th order polynomial). In most cases, use of the heuristic solution presented later is easier.

The q = 1 Family

The q = 1 family of the generalized, bivariate, linear location problem describes the median line that minimizes the absolute distance in the p-norm direction. The q = 1 family has an advantage in that the located linear facility is less susceptible to outliers. For q = 1 the objective function to be minimized is given by the following equation:

$$CL_p^1(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{(p-1)/p}}. \quad (4.37)$$

The following property implies a simplified solution technique for the q = 1 family.

<u>Property 4.3</u>

A solution (C_o^* and C_1^*) to Equation (4.37) exists which satisfies $a_{i1} = C_o^* + C_1^* a_{i1}$ for at least two existing facilities (i and k where $i \neq k$).

The following proof is a generalization of the method used by Morris and Norback (1980) for the p = 2 case.

<u>Proof</u>. Since a solution of the form $x_{i2} = C_o + C_1 x_{i1}$ exists, the axes can be translated without a loss of generality so that $C_o = 0$. Then $C\ell_p^1(C_1)$ can be written as follows:

$$C\ell_p^1(C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_1 a_{i1}|}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{(p-1)/p}}. \quad (4.38)$$

The existing facilities i can be renumbered so that the following region is specified:

$$R_r = \{C_1: \; a_{i2} - C_1 a_{i1} \geq 0, \; i = 1, \ldots, r;$$
$$a_{i2} - C_1 a_{i1} \leq 0, \; i = r + 1, \ldots, n\}. \quad (4.39)$$

The union of regions R_r, $r = 0, \ldots, n$ covers the real number line. When $r = 0$ ($r = n$) no values of i exist such that $a_{i2} - C_1 a_{i1} > 0$ ($a_{i2} - C_1 a_{i1} < 0$). For $C_1 \varepsilon R_r$ the absolute value sign can be removed from $C\ell_p^1(C_1)$ and the following equation can be written:

$$\begin{aligned} C\ell_p^1(C_1) &= [1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{-(p-1)/p} \\ &\quad * \{ \sum_{i=1}^{r} W_i (a_{i2} - C_1 a_{i1}) \\ &\quad - \sum_{i=r+1}^{n} W_i (a_{i2} - C_1 a_{i1}) \}, \\ &= [1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{-(p-1)/p} (A - C_1 B) \end{aligned} \quad (4.40)$$

where

$$A = \sum_{i=1}^{r} W_i a_{i2} - \sum_{i=r+1}^{n} W_i a_{i2},$$

$$B = \sum_{i=1}^{r} W_i a_{i1} - \sum_{i=r+1}^{n} W_i a_{i1}.$$

The term $(A - C_1B)$ in Equation (4.40) must be greater than zero unless the existing facilities are collinear, in which case the proof is trivial.

All that remains to be shown is that $C\ell_p^1(C_1)$ cannot be minimized on the interior of R_r. The first derivative of $C\ell_p^1(C_1)$ is as follows:

$$\frac{dC\ell_p^1(C_1)}{dC_1^*} = -[1 + [(C_1^2)^{1/2}]^{-[(2p-1)/p]}]$$
$$* (C_1[(C_1^2)^{1/2}]^{(2-p)/(p-1)} A + B). \qquad (4.41)$$

Setting Equation (4.41) equal to zero results in the following equations:

$$[(C_1^2)^{1/2}]^{(2-p)/(p-1)} C_1 = -[B/A] \qquad (4.42a)$$

or

$$C_1 = \pm[(B/A)^2]^{(p-1)/2}. \qquad (4.42b)$$

Taking the second derivative of Equation (4.41) produces the following derivative:

$$\frac{d^2C\ell_p^1(C_1)}{dC_1^2} = \frac{[(C_1^2)^{1/2}][p/(p-1)]-2}{(p-1)}$$
$$* [1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{-[(3p-1)/p]}$$
$$* [2(p-1)[(C_1^2)^{1/2}]^{p/(p-1)}A + (2p-1)$$
$$* C_1B - A]. \qquad (4.43)$$

The sign of $d^2C\ell_p^1(C_1)/dC_1^2$ is determined by the expression $[2(p-1)[(C_1^2)^{1/2}]^{p/(p-1)}A + (2p-1)C_1B - A]$. When the values of C_1 from Equation (4.42b) are substituted into the preceding expression, $d^2C\ell_p^1(C_1)/dC_1^2$ has the same sign as

$$2(p-1)[(B/A)^2]^{p/2}A \pm (2P-1)[(B/A)^2]^{(p-1)/2}B - A$$

$$= -\{\frac{[(B/A)^2]^{(p-2)/2}B^2 + A^2}{A}\}. \tag{4.44}$$

Thus, $d^2C\ell_p^1(C_1)/dC_1^2$ has the same sign as $-A$. $A - C_1^*B > 0$ is violated, however, when C_1 takes on the sign from Equation (4.42a) unless $A > 0$. Therefore, any extremum on the interior of R_r must be maximum when $p \neq 1$. Since $C\ell_p^1(C_1)$ is bounded below on R_r, $C\ell_p^1(C_1)$ is minimized on the boundary of R_r where at least one term of the form W_j: $a_{j2} - C_1^*a_{ji} = 0$.

Substituting C_1 in $C\ell_p^1(C_o, C_1)$, the condition function can be written as

$$C\ell_p^1(C_o|C_1) = \sum_{i=1}^{j-1} W_i'|a_{i2}' - C_o|$$

$$+ \sum_{i=j+1}^{n} W_i'|a_{i2}' - C_o| \tag{4.45}$$

where

$$W_i' = \frac{W_i}{[1 + [(C_1)^{1/2}]^{p/(p-1)}]^{(p-1)/p}},$$

$a_{i2}' = a_{i2} - C_1 a_{i1}$, and

the jth term of $C\ell_p^1(C_o, C_1)$ which equals zero is omitted.

$C\ell_p^1(C_o|C_1)$ is convex from Property 3.4 and piecewise linear with points of discontinuity in the first derivative occurring at a_{i2}', $i = 1, \ldots, n$; $i \neq j$. A minimizing value of C_o exists for which at least one term of the form $W_k|a_{k2}' - C_o| = 0$ and $j \neq k$. This condition means that at least the jth and kth terms of $C\ell_p^1(C_o, C_1)$ equal zero, and the property is proven for all p except $p = 1$.

The results of the previous property extend the simplistic technique developed by Morris and Norback (1980) for $p = 1$ and $q = 1$ to the general case of any value of p. To solve any $q = 1$ generalized, bivariate, linear location problem, determine the line that passes through two existing facilities and minimizes $C\ell_p^1(C_o, C_1)$.

The $p = \infty$ Family

The $p = \infty$ family of the generalized, bivariate, linear location problem is the special case where the error is divided equally between the two variables

x_1 and x_2. When $p = \infty$ (i.e., as with all cases when $p \neq 1$), the concept of independent and dependent variables no longer holds. Each variable in the $p = \infty$ model must be responsible for an equal share of the error introduced by the linear model. When $p = \infty$ the generalized, bivariate, linear location model becomes

$$C\ell_\infty^q = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + |C_1|]^q}. \qquad (4.46)$$

The distance being considered in this family is the 45-degree distance from the given facility to the linear facility to be located. This result is derived from Property 2.7 where the statement is made that the slope of the line between the given facility i and the linear facility to be located that minimizes the p-norm distance is given by $1/[(C_1^2)^{1/2}]^{1/(p-1)}$. When $p = \infty$ the slope is equal to 1. The Weber formulation of the $p = \infty$ family with the hyperbolic smoothing constant is as follows:

$$CL_\infty^q = \frac{\sum_{i=1}^{n} W_i [(a_{i2} - C_o - C_1 a_{i1})^2 + m]^{q/2}}{[1 + (C_1^2)^{1/2}]^q}. \qquad (4.47)$$

The special case of $p = \infty$ and $q = 2$ is interesting because of its close relationship to regression. If

If Equation (4.35) is rearranged, the following equation is obtained:

$$\{[\frac{\sum_{i=1}^{n} W_i[a_{i2}^* - C_1 a_{i1}^*]a_{i2}^*}{\sum_{i=1}^{n} W_i[a_{i2}^* - C_1 a_{i1}^*]a_{i1}^*}]^2\}^{1/2} = \frac{1}{[(C_1^2)^{1/2}]^{1/p-1}}. \quad (4.48)$$

If p approaches 1 from the negative direction as in the linear regression of x_2 on x_1, the limit of the term on the right side of the equality approaches infinity. The necessary and sufficient condition for the equality to hold is for the denominator of the term on the left side of the equality to approach zero. The denominator of the term on the left side of the equality set equal to zero is the normal equation for linear regression with mean-corrected data, and the term to the right of the equality is the slope of the line connecting the existing facility i to the point on the linear facility that minimizes the p distance.

A similar relationship holds when p approaches 1 from the positive direction. The limit of the term to the right of the equality is zero as the limit is taken as p approaches 1 from the positive direction. The left side of the equality can be equal to zero only when

$$\sum_{i=1}^{n} W_i[a^*_{i2} - C_1 a^*_{i1}]a^*_{i2} = 0.$$

As shown by Property 3.6, the preceding expression is the normal mean-corrected equation for the linear regression of x_1 on x_2.

If the limit of the right side of the equality in Equation (4.35) is taken as p approaches infinity, the following relationship results:

$$[\sum_{i=1}^{n} W_i[a^*_{i2} - C_1 a^*_{i1}]a^*_{i2}]^2$$

$$= [\sum_{i=1}^{n} W_i[a^*_{i2} - C_1 a^*_{i2} - C_1 a^*_{i1}]a^*_{i1}]^2. \quad (4.49)$$

The preceding equation results directly from the fact that the limit of $1/[(C_1)^{1/2}]^{1/(p-1)}$ is 1 as p approaches infinity. Equation (4.49) can be solved directly by expanding terms and using the quadratic formula. Thus, Equation (4.49) can be considered the normal equation for $p = \infty$, $q = 2$.

Note that values of p approaching 1 from the positive or negative direction and of p approaching infinity are the only cases of the generalized, bivariate, linear location problem where $1/[(C_1^2)^{1/2}]^{1/(p-1)}$ is a constant. For all other values of p, the line connecting the existing facility i to the point on the linear facility that minimizes the p distance is a function of the linear relationship (C_1).

The $0 < q < 1$ Family

The $0 < q < 1$ family of the generalized, bivariate, linear location problem deals with the general case where the existing facilities closest to the linear facility to be located have the greatest influence in the decision process. The problem of determining the location of the linear facility poses several difficulties in determining optimal solutions because of the nonconvexity, the nonconcavity, and the nondifferentiability of the objective function. Certain simplifications are available, however, that ease the computation difficulty.

As stated in Property 3.4, $L^q_{pi}(C_o, C_1)$ is strictly concave with respect to C_o for $0 < q < 1$ since the hyperbolic smoothing constant m is greater than zero. $C\ell^q_p(C_o, C_1)$, however, contains discontinuities in the first derivatives. These discontinuities allow for the proof which follows.

Property 4.4

Given the following formulation of $C\ell^q_p(C_o, C_1)$:

$$C\ell^q_p(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + [(C_1^2)^{1/2}]^{p/(p-1)}]^{q(p-1)p}}, \qquad (4.50)$$

an optimal minimizing solution exists for C_o and C_1 called C_o^* and C_1^* which satisfies $a_{i2} = C_o^* + C_1^* a_{i1}$ for at least one existing facility i when $0 < q < 1$.

<u>Proof</u>. For a proof by contradiction, first assume a solution exists for C_o, C_1 called C_o^* and C_1^* which minimizes Equation (4.50) and which does not satisfy $a_{i2} = C_o^* + C_1^* a_{i1}$ for any existing facility i. Then the existing facilities can be partitioned into two subsets depending upon whether $a_{i2} - C_o^* - C_1^* a_{i1}$ is greater than or less than zero. $C\ell_p^q(C_o^*, C_1^*)$ can then be rewritten within the absolute value signs as follows:

$$C\ell_p^q(C_o^*, C_1^*) = \frac{\sum_{i=1}^{k} W_i(a_{i2} - C_o^* - C_1^* a_{i1})^q}{[1 + [(C_1^{*2})^{1/2}]^{p/(p-1)}]^{q(p-1)/p}} + \frac{\sum_{i=k+1}^{n} W_i(C_o^* + C_1 a_{i1} - a_{i2})^q}{[1 + [(C_1^{*2})^{1/2}]^{p/(p-1)}]^{q(p-1)/p}} \tag{4.51}$$

where

$$0 < q < 1.$$

Given the value of the slope C_1^* that minimizes Equation (4.50), the conditional first and second derivatives for $C\ell_p^q(C_o^*|C_1^*)$ are as follows:

$$\frac{dC\ell_p^q(C_o^*|C_1)}{dC_o^*} = -q[1 + [(C_1^{*-2})^{1/2}]^{p/(p-1)}]^{-q(p-1)/p}$$

$$* \{ \sum_{i=1}^{k} W_i(a_{i2} - C_o^* - C_1^* a_{i1})^{q-1}$$

$$- \sum_{i=k+1}^{n} W_i(C_o^* + C_1^* a_{i1} - a_{i2})^{q-1} \}, \qquad (4.52)$$

$$\frac{d^2C\ell_p^q(C_o^*|C_1^*)}{dC_o^*} = q(q-1) \; [1 + [(C_1^{*2})^{1/2}]^{p/(p-1)}]^{-q(p-1)/p}$$

$$* \{ \sum_{i=1}^{k} W_i(a_{i2} - C_o^* - C_1^* a_{i2})^{q-2}$$

$$+ \sum_{i=k+1}^{n} W_i(C_o^* + C_1^* a_{i1} - a_{i2})^{q-2} \}. \qquad (4.53)$$

The second derivative Equation (4.53) is strictly less than zero when $q < 1$. Thus, C_o^* corresponds to a relative maximum rather than to a relative minimum. This condition requires that the minimizing linear facility pass through at least one existing facility, and the proof is complete.

The alternate proof mentioned earlier for the Barrodale and Roberts (1970) theorem that the linear facility minimizes the vertical distance where $0 < q < 1$ passes through at least two existing facilities is presented here.

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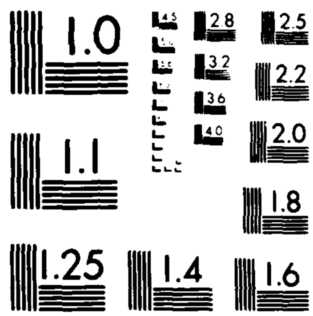

MICROCOPY RESOLUTION TEST CHART
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Property 4.5

Given the following formulation of $C\ell_{1-}^{q}(C_o, C_1)$:

$$C\ell_{1-}^{q}(C_o, C_1) = \sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q \tag{4.54}$$

where

$0 < q \leq 1,$

an optimal solution exists for C_o and C_1 called C_o^* and C_1^* that minimizes $C\ell_{1-}^{q}(C_o, C_1)$ and satisfies $a_{i2} = C_o^* + C_1^* a_{i2}$ for at least two existing facilities ($i = j$ and $i = k$ and $j \neq k$) when $0 < q < 1$.

Proof. From the results of Properties 4.3 and 4.4 the statement can be made that the linear facility that minimizes $C\ell_{1-}^{q}(C_o, C_1)$ will pass through at least one existing facility. Showing that the minimizing linear facility must pass through a second existing facility completes the proof. Using a method similar to the one used for Property 4.4, translate the coordinates so that the one existing facility (a_{i1}, a_{k2}) that is on the minimizing linear facility is at the origin. $C\ell_{1-}^{q}(C_o, C_1)$ can now be rewritten as:

$$C\ell_{1-}^{q}(C_1) = \sum_{i=1}^{k-1} W_i (a_{i2} - C_1 a_{i1})^q + \sum_{i=k+1}^{n} W_i (C_1 a_{i1} - a_{i2})^q$$

$$+ W_i (a_{k2} - C_1 a_{k1})^q \tag{4.55}$$

where

$i = 1, \ldots, k-1$ are the existing facilities that satisfy $a_{i2} - C_1 a_{i1} > 0$,

$i = k+1, \ldots, n$ are the existing facilities that satisfy $C_1 a_{i1} - a_{i2} > 0$, and

$W_k(a_{k2} - C_1 a_{k1})^q = 0.$

If the assumption is made that the value of C_1 called C_1^* for the linear facility that minimizes Equation (4.55) does not pass through a second existing facility, then $C\ell_{1-}^{q}(C_1)$ is both continuous and differentiable. Straightforward differentiation of Equation (4.55) results in the following equation:

$$\frac{dC\ell_{1-}^{q}(C_1)}{dC_1} = -q\left[\sum_{i=1}^{k-1} W_i(a_{i2} - C_1 a_{i1})^{q-1} - \sum_{i=k+1}^{n} W_i(C_1 a_{i1} - a_{i2})^{q-1}\right], \tag{4.56}$$

$$\frac{d^2C\ell_{1-}^{q}(C_1)}{dC_1^2} = q(q-1)\left[\sum_{i=1}^{k-1} W_i(a_{i2} - C_1 a_{i1})^{q-2} + \sum_{i=k+1}^{n} W_i(C_1 a_{i1} - a_{i2})^{q-2}\right]. \tag{4.57}$$

Since the second derivative is strictly negative for $0 < q < 1$, it follows that any C_1^* must be a relative maximum rather than a relative minimum as assumed. This contradiction requires that the relative minimum pass through a second existing facility, and the property is proven. (The proofs of Properties 4.4 and 4.5 are based on a method from Morris & Norback (1980) suggested by an anonymous referee.)

Property 4.5 can be easily extended to the limit as p approaches 1 from the positive direction by simply reversing the variables of the original problem. Thus, the horizontal distance of the original problem becomes the vertical distance of the modified problem. This process is similar to rotating the axis of the original problem by 90 degrees. Property 4.5 was extended to the $p = 2$ case by Morris and Norback (1980) when they rotated the original axis such that the optimal linear facility was horizontal. In the $p = 2$ case the vertical distance is the distance to be minimized since the Pythagorean theorem holds. For $p \neq 2$ or $p \neq 1$ the rotation method is not a valid solution technique because of the triangular inequality (Morris, 1981). However, the following property generalized from Morris and Norback (1980) eases the computational problems in determining an optimal solution for $p \neq 2$ or $p \neq 1$.

Property 4.6

Given the following formulation of $C\ell_p^q(C_o, C_1)$:

$$C\ell_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + |C_1|^{p/(p-1)}]^{q(p-1)/p}}, \qquad (4.58)$$

the following relationship holds:

$$|C_1^*|_v \le |C_1^*|_p \le |C_1^*|_h$$

where

C_{ov}^*, $|C_1^*|_v$ are the intercept and the absolute value of the optimal slope of the linear facility when p approaches 1 from the negative direction,

C_{op}^*, $|C_1^*|_p$ are the intercept and the absolute value of the optimal slope of the linear facility for any p, and

C_{oh}^*, $|C_1^*|_h$ are the intercept and the absolute value of the optimal slope of the linear facility when p approaches 1 from the positive direction.

Proof. From Equations (3.15), (3.16), and (4.58) the following equations can be stated:

$$C\ell_p^q(C_o,\ C_1) = [1 + |C_1|^{p/(p-1)}]^{-q(p-1)/p}$$

$$* \ C\ell_{1-}^q(C_o,\ C_1), \tag{4.59}$$

$$C\ell_p^q(C_o,\ C_1) = |C_1|\,[1 + |C_1|^{p/(p-1)}]^{-q(p-1)/p}$$

$$* \ C\ell_{1+}^q(C_o,\ C_1). \tag{4.60}$$

Now $C\ell_{1-}^q(C_{ov}^*,\ C_{1v}^*) \leq C\ell_{1-}^q(C_{op}^*,\ C_{1p}^*)$ since $(C_{ov}^*,\ C_{1v}^*)$ minimizes $C\ell_{1-}^q(C_o,\ C_1)$. So $[1+|C_1^*|_v^{p/(p-1)}]^{-q(p-1)/p}$ $* \ C\ell_{1-}^q(C_{ov}^*,\ C_{1v}^*) \leq [1 + |C_1^*|_v^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1-}^q(C_{op}^*,\ C_{1p}^*)$.

From Equation (4.45) and the previous expression, $C\ell_p^q(C_{op}^*,\ C_{1p}^*) \leq C\ell_p^q((C_{ov}^*,\ C_{1v}^*) = [1 + |C_1^*|_v^{p/(p-1)}]^{-q(p-1)/p}$ $* \ C\ell_{1-}^q(C_{ov}^*,\ C_{1v}^*)$. This means $[1 + |C_1^*|_p^{p/(p-1)}]^{-q(p-1)/p}$ $* \ C\ell_{1-}^q(C_{op}^*,\ C_{1p}^*) \leq [1 + |C_1|_v^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1-}^q(C_{ov}^*,\ C_{1v}^*)$ so that $|C_1^*|_v^{p/(p-1)} \leq |C_1^*|_p^{p/(p-1)}$. Similarly, $|C_1^*|_h [1 + |C_1^*|_h^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1+}^q(C_{oh}^*,\ C_{il}^*) \leq |C_1^*|_h$ $* \ [1 + |C_1^*|_h^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1+}^q(C_{op}^*,\ C_{1p}^*)$ and $|C_1^*|_p [1 + |C_1^*|_p^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1+}^q(C_{op},\ C_{1p}) \leq |C_1^*|_h$ $* \ [1 + |C_1^*|_h^{p/(p-1)}]^{-q(p-1)/p} C\ell_{1+}^q(C_{oh}^*,\ C_{1h}^*)$. Therefore, $|C_1^*|_p [1 + |C_1^*|_p^{p/(p-1)}]^{-q(p-1)/p} \leq |C_1^*|_h$ $* \ [1 + |C_1^*|_h^{p/(p-1)}]^{-q(p-1)/p}$ which resolves $|C_1^*|_p^{p/(p-1)} \leq |C_1^*|_h^{p/(p-1)}$ and completes the proof.

A general Heuristic solution for the $0 < q < 1$ family is to individually select each existing facility as the optimal C_o (i.e., from Property 4.4) and to vary C_1 between C^*_{1v} and C^*_{1h} for the slope that minimizes Equation (4.58). The optimal solution will be the existing facility that results in the minimum value for Equation (4.58).

The $q > 1$ Family

When $q > 1$ and the distance to be minimized is neither the vertical nor the horizontal distance, both $C\ell^q_p(C_o, C_1)$ and $CL^q_p(C_o, C_1)$ remain nonconvex and nonconcave. The only solution technique available under such circumstances is a two-dimensional *Heuristic search.* The search, however, is simplified as a result of Property 4.6 since the range for C^*_{1p} must be between C^*_{1v} and C^*_{1h}. The suggested procedure is to step through values of C_1 and to determine at each step the optimal value of C_o given the present value of C_1. C_o can be determined by any gradient search technique since $C\ell^q_p(C_o, C_1)$ is strictly convex. The optimal values for C_o and C_1 can then be determined by finding the minimum value of $C\ell^q_p(C_o, C_1)$.

Sample Problem

To compare the results of various combinations of p and q values for the generalized, bivariate, linear location problem, a sample problem selected from Neter and Wasserman (1974, p. 94) is presented. The optimal values of C_o and C_1 determined from 7 values of q ranging from .1 to infinity and from 6 values of p ranging from the vertical distance to the horizontal distance are calculated. The data points used are as follows:

i:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a_{i1}:	7	4	5	1	5	4	7	2	4	2	8	5	2	5	7	1	4	5
a_{i2}:	97	57	78	10	75	62	101	27	53	33	118	65	25	71	105	17	49	68
W_i:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The fact that the weights for each existing facility are equal to one does not restrict or limit the example. The results of the calculations are presented in Table 4.1.

Table 4.1

Results of Test Example

Values of q		Values of p					
		1^-	1^+	1.6	2.0	10	∞
.4	C_o =	-5.250#	-5.434#	-5.250*	-5.250#	-5.250*	-5.250*
	C_1 =	15.250	15.429	15.250	15.250	15.250	15.250
.7	C_o =	-5.250#	-5.427#	-5.429*	-5.429#	-5.249*	-5.429*
	C_1 =	15.450	15.427	15.429	15.429	15.429	15. 29
1.0	C_o =	-2.600#	-5.429#	-2.600#	-2.600#	-2.600#	-2.600#
	C_1 =	14.800	15.429	14.800	14.800	14.800	14.800
1.5	C_o =	-2.599°	-3.736°	-3.735*	-3.730*	-3.671*	-3.649*
	C_1 =	14.819	15.096	15.095	15.094	15.080	15.075
2.0	C_o =	-1.947	-3.160	-3.159*	-3.154	-3.102*	-3.083*
	C_1 =	14.693	14.973	14.973	14.972	14.961	14.955
3.0	C_o =	-1.516°	-2.653°	-2.652*	-2.647*	-2.595*	-2.576*
	C_1 =	14.579	14.845	14.845	14.844	14.832	14.827
∞	C_o =	-4.000°	-4.000°	-4.000*	-4.000*	-4.000*	-4.000*
	C_1 =	15.000	15.000	15.000	15.000	15.000	15.000

* indicates heuristic solutions.
\# indicates two point solutions.
° indicates gradient solutions.

CHAPTER V

IMPLICATIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

The derivation of the generalized, bivariate, linear location problem consists of two subproblems. The first subproblem involves the determination of the point on a linear facility that minimizes the p-norm distance to an individual existing facility. The second subproblem consists of determining the optimal linear facility that minimizes the sum of the q multiple of the p-norm distance from all the existing facilities to the point on the linear facility determined by the previous step. The simultaneous solution of the two subproblems results in the following formulation:

$$C\ell_p^q(C_o, C_1) = \frac{\sum_{i=1}^{n} W_i |a_{i2} - C_o - C_1 a_{i1}|^q}{[1 + |C_1|^{p/(p-1)}]^{q(p-1)/p}}.$$

The lack of convexity of the resulting formulation for the general case prohibits a universal solution technique of the generalized, bivariate, linear location problem for all combinations of possible values for p and q. For certain combinations of p's and q's an exact solution of generalized, bivariate, linear location problem can be determined. For the remaining combinations the only means of determining a solution is through a heuristic approximation procedure. In Table 5.1 the solution techniques available for each combination of p's and q's are outlined.

Implications

The generalized, bivariate, linear location problem expands the use of linear location theory to include the concept of a p-norm metric system of measurement. In previous linear location problems only the vertical distance (the limit as p approaches 1 from the positive direction) and the Euclidean distance ($p = 2$) were considered. The generalized, bivariate, linear location problem allows the analyst to place the linear facility while considering the importance of each coordinate axis. If the coordinates are measured in different units prohibiting conversion, for example, minimizing the Euclidean distance may not be consistent with reality since the $p = 2$ distance assumes an inverse importance of the axis depending upon the slope of the located facility. By proper selection of the p-norm distance to be minimized,

Table 5.1

Solution Techniques for the Generalized,

Bivariate, Linear Location Problem

Values of q	Values of p				
	$p = 1$*	$1 < p < 2$	$p = 2$	$2 < p < \infty$	$p = \infty$
$0 < q < 1$	D, F, G	F, G	D, F, G	F, G	F, G
$q = 1$	A, D, G	D, G	D, G	D, G	D, G
$1 < q < 2$	C, G	G	G	G	G
$q = 2$	A, B, C, E, G	E, G	B, E, G	E, G	B, E, G
$q > 2$	C, G	G	G	G	G
$2 < q < \infty$	C, G	G	G	G	G
$q = \infty$	A, C, G	G	G	G	G

Where

A = Linear Programming Formulation
B = Classical Optimization
C = Gradient Search, Convex Programming, and Kuhn-Tucker Conditions
D = Two Point Solution
E = Birge-Vieta Iterative Method
F = One Point Approximation Solution
G = General Heuristic Technique (Approximation)

*The limit as p approaches 1 from either the positive or from the negative direction.

the analyst may select the weighting of the coordinate system that best defines the importance between the two axes.

The generalized, bivariate, linear location problem can be extended, just as in Euclidean linear location theory and regression, to cases other than the minimization of the absolute or the square of the distance being considered. The value of q, which is the multiple of the absolute distance between the existing facility and the linear facility to be located, can be varied between greater but not equal to zero and infinity. For the most part, the solution techniques available for linear regression extend to the generalized, bivariate, linear location problem when q is less than or equal to one. In general, however, for q greater than one the generalized, bivariate, linear location problem must depend upon a heuristic search procedure that does not guarantee an optimal solution.

Recommendations for Further Research

Three areas should be considered for future research concerning the generalized, bivariate, linear location problem.

1. The first area concerns the problem of determining the optimal combination of p and q values that

should be used in any one problem. This problem may be of a subjective nature in most situations. If the models presented in this research are used in a forecasting situation, however, the problem becomes quite objective since the desire is to obtain the best fit.

2. A second and obvious extension of the bivariate problem is the multivariate case. A more obscure extension is the multivariate problem where the p values for each variable are not equal. The problem can be formulated as follows:

$$C\ell^{q}_{p_1,\, p_2,\, \ldots,\, p_{r+1}}(C_o,\, C_1,\, \ldots,\, C_r)$$

$$= \frac{\sum_{i=1}^{n} W_i |a_{i,r+1} - C_o - C_1 a_{i1} \ldots, C_r a_{ir}|^q}{[1 + |C_1|^{p_1/(p_1-1)} + \ldots, + |C_r|^{p_r/(p_r-1)}]^{q(p_{r+1}-1)/p_r}}.$$

3. The third area concerns the statistical applications of the generalized, bivariate, linear location problem. Extensive work must be done to provide sampling distributions for the various parameters of the model.

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BIOGRAPHICAL SKETCH

Joseph William Coleman was born in New Castle, Pennsylvania, on April 13, 1945. He received his elementary and secondary education in the New Castle Public Schools. In 1963 he entered the Pennsylvania State University, graduating in 1967 with a Bachelor of Science with a major in Electrical Engineering. He entered the United States Air Force in 1968 and served at the National Security Agency from 1968 to 1972 and Vandenberg Air Force Base, California, from 1972 to 1977. In May, 1975, he was awarded a Masters of Business Administration from Golden Gate University. Since July, 1980, he has been on the faculty of the United States Air Force Institute of Technology at Wright-Patterson Air Force Base, Ohio. He is a member of the Alpha Iota Delta Honorary Society and The Institute of Management Sciences. He is married to the former Florence Ella Squire and the father of one daughter, Joy Lynn.